

Smoking dynamics with media awareness to control the prevalence of bad effect through fractional operator study

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ABSTRACT

Smoking remains a persistent global health concern with complex behavioral dynamics influenced by memory and past experiences. This study formulates and analyses a fractional-order mathematical model of smoking behavior using the Caputo derivative to capture memory effects and non-local interactions. The well-posedness of the model is ensured through rigorous proofs of existence and uniqueness of solutions. To assess the system's resilience, Hyers-Ulam-Rassias stability is investigated under small perturbations. To address potential chaotic behavior, we implement chaos control techniques, stabilizing the system for reliable long-term predictions. A novel Newton polynomial-based numerical scheme is developed to efficiently approximate solutions, validated through extensive simulations. Our results demonstrate that fractional-order modeling provides deeper insights into smoking dynamics compared to classical approaches. Some key features of the proposed method include:

- Investigating Hyers-Ulam-Rassias stability to analyze robustness against perturbations.
- Applying chaos control techniques to manage and stabilize chaotic system behavior.
- Developing and implementing a Newton polynomial-based numerical scheme for efficient solution approximation.

Specifications table

Subject area	Mathematics and Statistics
More specific subject area	Applied mathematics
Name of your method	Chaos control and Newton polynomial
Name and reference of original method	None
Resource availability	None

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Background

The increasing death rate of tobacco users worldwide is proof that tobacco use is turning into a serious addiction that needs to be controlled globally. Many people started smoking while they were young and were unaware of the risks. Smoking needs to be considered a serious issue on a worldwide basis to stop the development of smoking behaviors. The control of smoking addiction is significantly impacted by the media. It makes people more conscious of the negative consequences of smoking. Regular smoking commercials have the power to persuade smokers to alter their smoking behaviors. Cigarette smoking results in a gradual and excruciating death. It is at the root of many deadly diseases. The media is crucial in at least disseminating helpful information about the negative effects of smoking. Among other things, it has the ability to directly communicate to people about quitting smoking and altering their attitudes, beliefs, and behavior. To achieve the long-term objective of reducing tobacco use, a variety of all-encompassing programs have been put forth to entice smokers to attempt quitting. Depending on their length and level of intensity, mass media campaigns may or may not be successful in encouraging people to quit smoking. Smoking taxes, mass media advertising campaigns, peer education programs, community mobilization, motivational interviews, marketing restrictions, tobacco product health warnings, and public smoking bans are just a few of the numerous strategies the government has been using. While media and awareness campaigns are important in helping some members of society avoid substance misuse or achieve the smoking cessation stage, people who are struggling with stress, anxiety, depression, and other related issues are disproportionately affected by these efforts. These people are readily swayed by smokers and adopt this habit to feel better temporarily. (NIHDrugAbuseReport) highlights this feature. The environment has an impact on the chemical composition of tobacco. These leaves contain a variety of compounds. Several compounds, including benzopyrene, Nicotine-derived nitrosamine ketone, hydrogen cyanide, carbon monoxide, phenol, aldehydes, nicotine, and harmful alkaloids, are found in tobacco smoke. Tobacco also contains the radioactive element polonium 210. The primary cause of cancer is tobacco, which contains about 4000 harmful chemicals [1,2]. The chemical composition of smoke is influenced by aspects such as puff frequency. Nicotine is the main cause of higher blood pressure, heart rate, nervous system disruption, and the loss of the small blood vessels that underlie wrinkles. Carbon monoxide (CO) causes the lungs oxygen content to fall. The microscopic hairs that act as a natural lung cleaner are killed by hydrogen cyanide. Lead, arsenic, nickel, and cadmium are also present in smoke. Additionally, smoke carries herbicides like Dichloro-Diphenyl-Trichloroethane. Smoke contains a harmful chemical that is the main cause of lung and skin cancer. According to [3], smoking will cause 10 million fatalities in the 20th century and 1 billion in the 21st. Cigarette smoking has a negative impact on human fertility [4].

In [5], numerous applications of mathematical modeling are discussed. The same study describes how tobacco use affects the prevalence of tuberculosis, including an optimal control study. This study unequivocally demonstrates that tobacco-related measures can reduce the number of TB infections. Due to their increased susceptibility to TB compared to nonsmokers, smoking behaviors are an important factor that can increase TB transmission rates [6]. This is because smoking harms the lungs and impairs immunity, increasing a person's susceptibility to infection with *Mycobacterium tuberculosis*. A study on related concepts may be found in [7]. The authors examined a smoking cessation model with a media campaign in the study [8]. The findings showed that the reproduction number is suppressed when media campaigns aimed at quitting smoking are strengthened. Publicizing information to encourage smokers to stop was therefore a successful intervention. Fractional calculus mathematicians introduced fractional geometry, fractional dynamics, and fractional differential equations in the 19th century. Almost every scientific discipline makes use of fractional calculus. It is employed to simulate both engineering and physical processes. Standard integer-order mathematical models frequently fail to function as intended. As a result, fractional calculus significantly advanced the fields of image processing, chemistry, biology, and mechanics. Some recent studies are the role of fractional operators and Ulam-Hyers stability in understanding COVID-19 dynamics [9] and the long-term impacts of vaccination strategies. In study, [10] employed the Atangana-Baleanu operator in a fractional-order model assessing long-term cancer risk from persistent smoking, while [11] analyzed an age-structured smoking model with government intervention using fractal-fractional derivatives. The study [12] develops and compares mathematical models to predict daily confirmed COVID-19 cases. It provides valuable insights into model fitting and forecasting accuracy for pandemic trend analysis. Fractional stochastic models have gained attention for their ability to capture memory and randomness in epidemic dynamics. Building on this framework, a constant proportional Caputo operator was later introduced to model childhood disease epidemics, offering a more flexible and memory-sensitive approach to better reflect the underlying biological processes [13]. The SEI, I_s, R model with different infection stages [14], employing fractional operators to reveal nuanced dynamics and transmission patterns in disease modeling. A fractional-order predator-prey model incorporating memory effects and environmental fluctuations [15], demonstrating the relevance of fractional derivatives in ecological interactions. In particular, [16] examined the existence and Ulam stability of solutions for equations involving mixed Caputo-Riemann derivatives using fixed point theory. Similarly, [17] established the existence and uniqueness of solutions for nonlinear two-point boundary value problems involving the Caputo fractional derivative. Furthermore, [18] explored the controllability of Sobolev-type fractional integro-differential systems within a Banach space framework, contributing to the advancement of fractional dynamic system theory. In addition, [19] formulated mathematical models to assess the impacts of global warming and dust pollutants on the progression of desertification, offering predictive tools for mitigating environmental degradation through quantitative analysis [20]. developed a generalized cholera transmission model that incorporates asymptomatic individuals, enhancing early detection capabilities and informing more effective control strategies. In addressing digital behavior, [21] proposed a mathematical model with optimal control mechanisms to mitigate TikTok addiction, offering policy-relevant insights for managing excessive social media use. Additionally, [22] performed a numerical investigation of a typhoid transmission model within a fuzzy environment, demonstrating the impact of parameter uncertainty on disease dynamics and intervention effectiveness. The fractional-order SIRS epidemic model with a Caputo operator [23], provides a deeper understanding of disease transmission under varying immunity levels. The use of a fractional optimal control approach in epidemiological modeling [24], optimizing intervention

strategies and assessing the long-term effectiveness of public health policies. The study [25] analyzes and simulates a fractional-order smoking epidemic model, highlighting the influence of fractional derivatives on the dynamics of smoking spread and cessation strategies. The novel technique to analyze the solutions of a time-fractional nonlinear smoking epidemic model [26], provides deeper insights into its complex dynamics and long-term behavior.

Smoking remains a major global health challenge, causing millions of deaths annually. Traditional models often fail to capture the memory effects and complex dynamics of smoking behavior. This study introduces a fractional-order model using the Caputo derivative to incorporate historical influences like media exposure and addiction relapse. By modeling transitions among smoking-related groups and applying chaos control and a Newton polynomial-based numerical scheme, it offers a more accurate, stable, and efficient tool. The findings provide actionable insights for policymakers to design effective interventions, particularly in regions with high smoking rates. The Caputo derivative was specifically chosen because it allows the use of traditional initial conditions, which are more physically interpretable in real-world models like smoking dynamics. It effectively captures memory effects and hereditary properties in behavioral processes, providing a more realistic framework than classical or other fractional derivatives. Additionally, it supports well-established analytical and numerical techniques suitable for complex systems. This study introduces a novel approach to modeling smoking dynamics by employing the Caputo fractional derivative, which captures memory effects and non-local interactions more accurately than traditional integer-order models. It stands out by investigating Hyers-Ulam-Rassias stability to analyze the system's robustness against perturbations, a unique contribution to smoking behavior research. The application of chaos control techniques further enhances the model's practicality by stabilizing chaotic dynamics in smoking prevalence. Additionally, the development of a Newton polynomial-based numerical scheme provides an efficient method for approximating solutions to the fractional-order system. The inclusion of media awareness as a dynamic compartment adds realism, reflecting the impact of public health campaigns on smoking behavior. Together, these advancements offer deeper insights into smoking dynamics and improve the framework for evaluating intervention strategies.

The manuscript is organized as follows: section 2 outlines the methods, including the fractional operator utilized in the proposed model. It provides details about the compartments, parameters, and in-depth analysis of the proposed model. This is followed by the method, which examines the Hyers-Ulam-Rassias stability, and investigates the behavior of the proposed model under chaotic conditions within a feasible range. Section 2 validates the method by presenting numerical results, and providing a comprehensive discussion of the results.

Method details

Basic concepts

A few important definitions that may be useful for system analysis are given below.

Definition 1. [27] Considering a continuous function $B(t)$, the definition of a Caputo operator of order ς , where $c - 1 < \varsigma = 1$, is

$${}_0^C D_t^\varsigma B(t) = \frac{1}{\Gamma(c - \varsigma)} \int_0^t (t - \xi)^{c-\varsigma-1} [B'(\xi)] d\xi. \quad (1)$$

Definition 2. [27] The Riemann Liouville integration with respect to t is defined as follows when $B(t)$ is recognized as a continuous function

$${}_0^R I_t^\varsigma B(t) = \frac{1}{\Gamma(\varsigma)} \int_0^t (t - \xi)^{\varsigma-1} B(\xi) d\xi, \quad \varsigma > 0, \quad (2)$$

with a converging integral.

Formulation of model

The entire population under consideration is divided into five compartments in the study [28]: Individuals who currently smoke (S_i), Former smokers who have quit but may relapse (S_q), Individuals at risk of starting smoking (S_p), Individuals influenced by media and awareness who are less likely to start smoking (S_m), and the media compartment M , which is the additional efforts made by smokers to eradicate the prevalence of smoking in the population through media campaigns. It is believed that interacting with smokers can lead to those in the potential smoker class starting to smoke. Furthermore, those who are at risk of smoking but are not yet addicted to the habit proceed to the awareness media section after learning about the negative consequences of smoking. Those who give up smoking go to the quitter compartment. Quitters may relapse and start smoking again. Over time, those who are aware may develop into potential smokers as a waning may take place. The mortality rate from smoking-related causes in smoker and quitter groups. The media compartment M in this model represents extra media efforts measured by the total number of print, radio, television, internet, and other media advertisements. Here, ϕ and m_0 are constants that represent the depletion rate in media campaigns and the density of media campaigns among smokers, respectively. Table 1 describes the model parameters, and Fig. 1 shows the schematic diagram of the framed model.

To improve the earlier model, a fractional-order mathematical model is proposed in the classical Caputo sense. The model uses Caputo fractional derivatives to account for memory effects and includes transitions between compartments based on interaction

Table 1
Description of parameters and its values.

Symbols	Description of parameters [28]	Values
Λ	Rate of recruitment	2×10^1
μ	Natural Death Rate	5.479×10^{-6}
δ_1	Smokers death rate due to smoking related disease	3×10^{-2}
β	Potential-to-smoker conversion rate.	5×10^{-2}
ν	Quitters-to-smoker relapse rate	6×10^{-2}
γ	Smoker-to-quitter transition rate	1×10^{-2}
δ_2	Death rate of quitters due to smoking related disease	1×10^{-2}
ϵ	Awareness rate keeping potential smokers in media compartment	2×10^{-4}
ϵ_1	Awareness-to-potential smoker transition rate	2×10^{-2}
ϕ	Smoker-dependent media campaign addition rate.	7×10^{-4}
m_0	Decay rate of media campaigns	6×10^{-2}

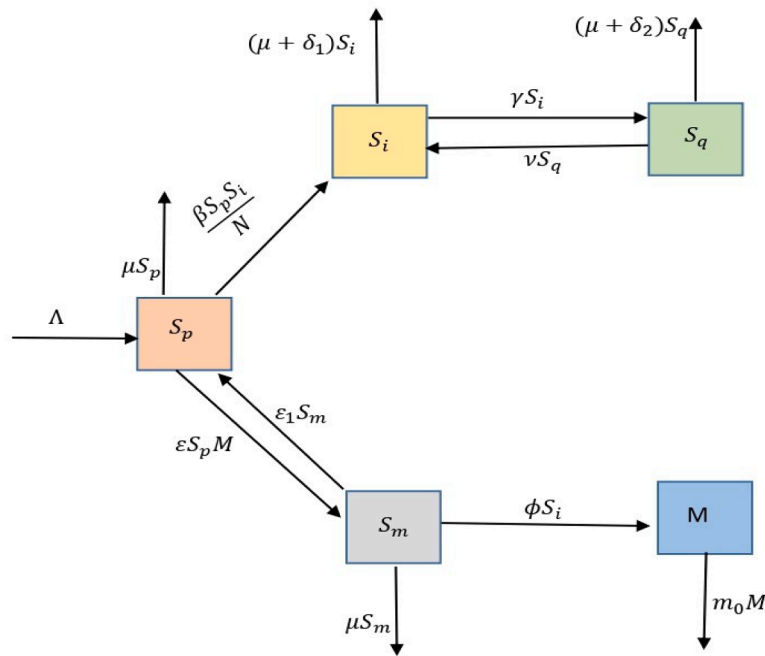


Fig. 1. Schematic diagram of the model.

rates, awareness efforts, relapse behavior, and natural or smoking-related mortality. This structure enables a dynamic and realistic simulation of long-term smoking trends influenced by behavior, memory, and media. The model is expressed as follows

$$\begin{cases} {}^C D_t^\epsilon S_p(t) = \Lambda - \left(\frac{\beta S_p S_i}{N} \right) - \mu S_p + \epsilon_1 S_m - \epsilon S_p M, \\ {}^C D_t^\epsilon S_i(t) = \left(\frac{\beta S_p S_i}{N} \right) - (\mu + \delta_1 + \gamma) S_i + \nu S_q, \\ {}^C D_t^\epsilon S_q(t) = \gamma S_i - (\mu + \delta_2) S_q - \nu S_q, \\ {}^C D_t^\epsilon S_m(t) = \epsilon S_p M - \mu S_m - \epsilon_1 S_m, \\ {}^C D_t^\epsilon M(t) = \phi S_i - m_0 M, \end{cases} \quad (3)$$

in terms of the initial conditions

$$S_p(0) \geq 0, \quad S_i(0) \geq 0, \quad S_q(0) \geq 0, \quad S_m(0) \geq 0, \quad M(0) \geq 0.$$

Positively invariant region

This section demonstrates the closed set

$$\theta = \{ (S_p, S_i, S_q, S_m, M) \in \mathbb{R}_+^5 : 0 \leq \mathbb{N} \},$$

provides the system (3) positively invariant feasibility region.

Lemma 1. Solutions of the system (3) starting at θ are non-negative and bounded.

Proof. The model (3) leads us to find that

$$\begin{cases} {}^C D_t^\epsilon S_p(t) \Big|_{S_p=0} = \Lambda + \varepsilon_1 S_m \geq 0, \\ {}^C D_t^\epsilon S_i(t) \Big|_{S_i=0} = \nu S_q \geq 0, \\ {}^C D_t^\epsilon S_q(t) \Big|_{S_q=0} = \gamma S_i \geq 0, \\ {}^C D_t^\epsilon S_m(t) \Big|_{S_m=0} = \varepsilon S_p M \geq 0, \\ {}^C D_t^\epsilon M(t) \Big|_{M=0} = \phi S_i \geq 0. \end{cases} \quad (4)$$

All rates are non-negative on the bounding planes in this situation. Consequently, the trajectory will remain within the non-negative bounded cone $\mathbb{R}_+^5$ if it begins in its interior region. This is ensured by the inward orientation of the vector field on all boundary planes. As a result, the solutions to (3) remain non-negative. Furthermore, based on the model presented in (3), it can be concluded that the total population, denoted as N , satisfies certain conditions

$${}^C D_t^\epsilon N(t) = \Lambda - \mu - \delta_1 S_i - \delta_2 S_q.$$

This gives

$$\lim_{t \rightarrow \infty} \sup N \leq \frac{\Lambda}{\mu}.$$

Therefore, every solution S_p , S_i , S_q and S_m is bounded by $\frac{\Lambda}{\mu}$. This provides the system's (2.1) biologically feasible region using the positively invariant set below

$$\theta = \left\{ (S_p, S_i, S_q, S_m, M) \in \mathbb{R}_+^5 : 0 \leq S_p, S_i, S_q, S_m \leq \frac{\Lambda}{\mu}, 0 \leq M \leq \frac{\phi}{m_0} = M_R \right\},$$

which is the region of attraction for the system. $\square$

Equilibrium point analysis

To determine the equilibrium points of the fractional-order system, the fractional derivatives must be set to zero, as equilibrium implies that the state variables remain constant over time. Then the equilibrium points of the model (3) are

$$E^* = (S_p^*, S_i^*, S_q^*, S_m^*, M) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0 \right). \quad (5)$$

Reproduction number

The basic reproduction number R_0 is the average number of new smokers that a single smoker produces over the course of their whole smoking lifetime in a population of potential smokers. The Next-Generation Matrix technique [29] is employed to calculate the basic reproduction number R_0 . The matrix F and V are calculated as follows:

$$F = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} (\mu + \gamma + \delta_1) & -\nu \\ -\gamma & (\mu + \delta_2 + \nu) \end{bmatrix},$$

R_0 is the basic reproduction number, which is derived from the spectral radius of the matrix FV^{-1} .

$$R_0 = \frac{\beta(\mu + \delta_2 + \nu)}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu}. \quad (6)$$

Sensitivity analysis

The sensitivity analysis identifies the most influential parameters affecting smoking prevalence and highlights the critical role of media awareness in mitigating harmful effects within the fractional-order model. The influence of each parameter on the smoking dynamics model is illustrated in Fig. (2) through sensitivity analysis.

$$\frac{\partial R}{\partial \beta} \times \frac{\beta}{R} = \frac{\mu + \delta_2 + \nu}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu} > 0,$$

$$\frac{\partial R}{\partial \mu} \times \frac{\mu}{R} = \frac{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu \cdot \beta - \beta(\mu + \delta_2 + \nu)(2\mu + \gamma + \nu + 2\delta_2)}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu^2} < 0,$$

$$\frac{\partial R}{\partial \delta_2} \times \frac{\delta_2}{R} = \frac{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu \cdot \beta - \beta(\mu + \delta_2 + \nu)(2\delta_2 + 2\mu + \nu + \gamma)}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma\nu^2} < 0,$$

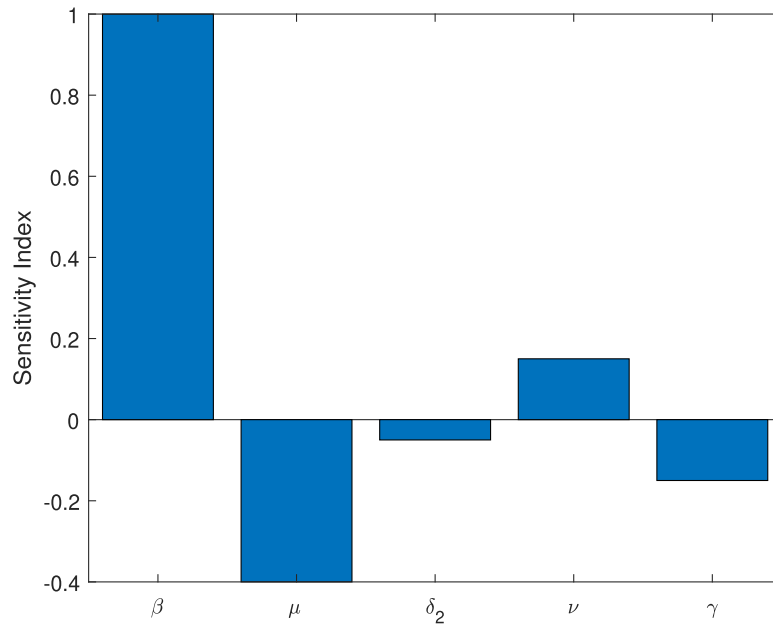


Fig. 2. Sensitivity analysis showing the impact of key parameters.

$$\frac{\partial R}{\partial \nu} \times \frac{\nu}{R} = \frac{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma \nu \cdot \beta - \beta(\mu + \delta_2 + \nu)(\mu + \delta_2)}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma \nu^2} > 0,$$

$$\frac{\partial R}{\partial \gamma} \times \frac{\gamma}{R} = \frac{-\beta(\mu + \delta_2 + \nu)(\mu + \delta_2)}{(\mu + \gamma + \delta_2)(\mu + \nu + \delta_2) - \gamma \nu^2} < 0.$$

Existence and uniqueness

This section will analyze the existence, uniqueness, and stability of the model (3) as well as the Lipchitz condition. In the analysis of existence and uniqueness, the Banach space of continuous functions $C([0, T], \mathbb{R}^5)$, endowed with the standard supremum norm $\|x\| = \sup_{t \in [0, T]} \|x(t)\|$, is employed. This norm provides the necessary framework to validate the Lipschitz condition and apply fixed point theorems. The normed space ensures that the solution mapping remains contractive under the conditions specified. Analyzing the existence and uniqueness of fractional-order models using the Caputo derivative is crucial to understanding smoking dynamics. Therefore, in order to keep things simple and clear, the analysis begins by considering the following five kernels

$$\begin{aligned} {}^C D_{0,t}^\varsigma S_p(t) &= \lambda_1(t, S_p), \\ {}^C D_{0,t}^\varsigma S_i(t) &= \lambda_2(t, S_i), \\ {}^C D_{0,t}^\varsigma S_q(t) &= \lambda_3(t, S_q), \\ {}^C D_{0,t}^\varsigma S_m(t) &= \lambda_4(t, S_m), \\ {}^C D_{0,t}^\varsigma M(t) &= \lambda_5(t, M). \end{aligned} \quad (7)$$

The above equation (7) can be solved by using the fractional-order Caputo operator.

$$\begin{aligned} S_p(t) - S_p(0) &= \wp(\varsigma) \lambda_1(t, S_p) + \wp(\varsigma) \int_0^t \lambda_1(\xi, S_p) d\xi, \\ S_i(t) - S_i(0) &= \wp(\varsigma) \lambda_2(t, S_i) + \wp(\varsigma) \int_0^t \lambda_2(\xi, S_i) d\xi, \\ S_q(t) - S_q(0) &= \wp(\varsigma) \lambda_3(t, S_q) + \wp(\varsigma) \int_0^t \lambda_3(\xi, S_q) d\xi, \end{aligned} \quad (8)$$

$$S_m(t) - S_m(0) = \wp(\varsigma)\lambda_4(t, S_m) + \varphi(\varsigma) \int_0^t \lambda_4(\xi, S_m) d\xi,$$

$$M(t) - M(0) = \wp(\varsigma)\lambda_5(t, M) + \varphi(\varsigma) \int_0^t \lambda_5(\xi, M) d\xi,$$

$\wp(\varsigma)$ and $\varphi(\varsigma)$ are positive real constants. Next, the Lipschitz condition of the Caputo system as stated in (3) can be confirmed.

Theorem 1. The preceding kernels, $\lambda_1(t, S_p)$, $\lambda_2(t, S_i)$, $\lambda_3(t, S_q)$, $\lambda_5(t, S_m)$, and $\lambda_5(t, M)$, all satisfy the Lipchitz requirement.

Proof. The first is that the λ_1 kernel justifies the Lipchitz requirement. The comparable norm, considering two functions, S_p and $\bar{S}_p$, is as follows

$$\begin{aligned} & \left\| \lambda_1(t, S_p) - \lambda_1(t, \bar{S}_p) \right\| \\ & \leq \left\| \left(-\frac{\beta S_p S_i}{N} - \mu S_p - \varepsilon S_p M \right) - \left(-\frac{\beta \bar{S}_p S_i}{N} - \mu \bar{S}_p - \varepsilon \bar{S}_p M \right) \right\| \\ & \leq \left(\frac{\beta S_i}{N} + \mu + \varepsilon M \right) \|S_p - \bar{S}_p\| \\ & \leq \chi_1 \|S_p - \bar{S}_p\|, \end{aligned} \quad (9)$$

taking $\chi_1 = \frac{\beta S_i}{N} + \mu + \varepsilon M$, where $\frac{\beta S_i}{N}$, μ and εM are bounded functions. The norms for the other model equations can be established in a similar way. It will be demonstrated that the Caputo model (3) under examination has at least one solution. Therefore, the circular formula for (8) is as follows

$$\begin{cases} S_p^n(t) = \wp(\varsigma)\lambda_1(t, S_p^{n-1}) + \varphi(\varsigma) \int_0^t \lambda_1(\xi, S_p^{n-1}) d\xi, \\ S_i^n(t) = \wp(\varsigma)\lambda_2(t, S_i^{n-1}) + \varphi(\varsigma) \int_0^t \lambda_2(\xi, S_i^{n-1}) d\xi, \\ S_q^n(t) = \wp(\varsigma)\lambda_3(t, S_q^{n-1}) + \varphi(\varsigma) \int_0^t \lambda_3(\xi, S_q^{n-1}) d\xi, \\ S_m^n(t) = \wp(\varsigma)\lambda_4(t, S_m^{n-1}) + \varphi(\varsigma) \int_0^t \lambda_4(\xi, S_m^{n-1}) d\xi, \\ M^n(t) = \wp(\varsigma)\lambda_5(t, M^{n-1}) + \varphi(\varsigma) \int_0^t \lambda_5(\xi, M^{n-1}) d\xi. \end{cases} \quad (10)$$

The initial conditions that are positive are the first iteration values. The following are the differences between two terms that follow one another.

$$\begin{aligned} \Psi_1^n &= S_p^n(t) - S_p^{n-1}(t) \\ &= \wp(\varsigma) \left(\lambda_1(t, S_p^{n-1}) - \lambda_1(t, S_p^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_1(\xi, S_p^{n-1}) - \lambda_1(\xi, S_p^{n-2}) \right) d\xi, \end{aligned} \quad (11)$$

$$\begin{aligned} \Psi_2^n &= S_i^n(t) - S_i^{n-1}(t) \\ &= \wp(\varsigma) \left(\lambda_2(t, S_i^{n-1}) - \lambda_2(t, S_i^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_2(\xi, S_i^{n-1}) - \lambda_2(\xi, S_i^{n-2}) \right) d\xi, \end{aligned} \quad (12)$$

$$\begin{aligned} \Psi_3^n &= S_q^n(t) - S_q^{n-1}(t) \\ &= \wp(\varsigma) \left(\lambda_3(t, S_q^{n-1}) - \lambda_3(t, S_q^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_3(\xi, S_q^{n-1}) - \lambda_3(\xi, S_q^{n-2}) \right) d\xi, \end{aligned} \quad (13)$$

$$\begin{aligned} \Psi_4^n &= S_m^n(t) - S_m^{n-1}(t) \\ &= \wp(\varsigma) \left(\lambda_4(t, S_m^{n-1}) - \lambda_4(t, S_m^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_4(\xi, S_m^{n-1}) - \lambda_4(\xi, S_m^{n-2}) \right) d\xi, \end{aligned} \quad (14)$$

$$\begin{aligned}\Psi_5^n &= M^n(t) - M^{n-1}(t) \\ &= \wp(\varsigma)(\lambda_5(t, M^{n-1}) - \lambda_5(t, M^{n-2})) + \varphi(\varsigma) \int_0^t (\lambda_5(\xi, M^{n-1}) - \lambda_5(\xi, M^{n-2})) d\xi.\end{aligned}\quad (15)$$

It's crucial

$$\sum_{i=0}^n \Psi_1^i = S_p^n(t), \sum_{i=0}^n \Psi_2^i = S_i^n(t), \sum_{i=0}^n \Psi_3^i = S_q^n(t), \sum_{i=0}^n \Psi_4^i = S_m^n(t), \sum_{i=0}^n \Psi_5^i = M^n(t). \quad (16)$$

The following analysis is conducted using the system of equations (11)-(15).

$$\begin{aligned}\Psi_1^n &= \|S_p^n(t) - S_p^{n-1}(t)\| \\ &= \left\| \wp(\varsigma) \left(\lambda_1(t, S_p^{n-1}) - \lambda_1(t, S_p^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_1(\xi, S_p^{n-1}) - \lambda_1(\xi, S_p^{n-2}) \right) d\xi \right\|.\end{aligned}$$

The equation above is reduced using the triangle inequality to

$$\begin{aligned}\|S_p^n(t) - S_p^{n-1}(t)\| &\leq \wp(\varsigma) \left\| \left(\lambda_1(t, S_p^{n-1}) - \lambda_1(t, S_p^{n-2}) \right) \right\| + \varphi(\varsigma) \left\| \int_0^t \left(\lambda_1(\xi, S_p^{n-1}) - \lambda_1(\xi, S_p^{n-2}) \right) d\xi \right\|.\end{aligned}$$

The kernel $\lambda_1(t, S_p)$ is shown to satisfy the Lipchitz criterion by Eq. (8). Thus, it can be written as

$$\|S_p^n(t) - S_p^{n-1}(t)\| \leq \wp(\varsigma) \chi_1 \|S_p^{n-1} - S_p^{n-2}\| + \varphi(\varsigma) \chi_1 \int_0^t \|S_p^{n-1} - S_p^{n-2}\| d\xi.$$

The previously described inequality can be reduced as follows using the data in (16).

$$\|\Psi_1^n(t)\| \leq \wp(\varsigma) \chi_1 \|\chi_1^{(n-1)}(t)\| + \varphi(\varsigma) \chi_1 \int_0^t \|\Psi_1^{(n-1)}(\xi)\| d\xi. \quad (17)$$

Similarly, the following findings are obtained

$$\begin{aligned}\|\Psi_2^n(t)\| &\leq \wp(\varsigma) \chi_2 \|\Psi_2^{(n-1)}(t)\| + \varphi(\varsigma) \chi_2 \int_0^t \|\Psi_2^{(n-1)}(\xi)\| d\xi, \\ \|\Psi_3^n(t)\| &\leq \wp(\varsigma) \chi_3 \|\Psi_3^{(n-1)}(t)\| + \varphi(\varsigma) \chi_3 \int_0^t \|\Psi_3^{(n-1)}(\xi)\| d\xi, \\ \|\Psi_4^n(t)\| &\leq \wp(\varsigma) \chi_4 \|\Psi_4^{(n-1)}(t)\| + \varphi(\varsigma) \chi_4 \int_0^t \|\Psi_4^{(n-1)}(\xi)\| d\xi, \\ \|\Psi_5^n(t)\| &\leq \wp(\varsigma) \chi_5 \|\Psi_5^{(n-1)}(t)\| + \varphi(\varsigma) \chi_5 \int_0^t \|\Psi_5^{(n-1)}(\xi)\| d\xi.\end{aligned}\quad (18)$$

□

Theorem 2. The fractional-order Caputo model (3) has requirements that permit an analytical solution at Ψ_0 :

$$\wp(\varsigma) \chi_i + \varphi(\varsigma) \chi_i \Psi_{1,0} < 1, \quad \text{for } i = 1, 2, \dots, 5.$$

Proof. It is observed that the functions $S_p(t)$, $S_i(t)$, $S_q(t)$, $S_m(t)$, and $M(t)$ are bounded, and the Lipschitz condition is satisfied due to the recursive relation and Eq. (17).

$$\|\Psi_1^n\| \leq \|S_p(0)\| [\wp(\varsigma) \chi_i + \varphi(\varsigma) \chi_i t]^n.$$

As a result, the previously mentioned solutions will remain in use. Yet to show that the functions listed above represent the suggested model solution, considering

$$S_p(t) - S_p(0) = S_p^n(t) - \Psi_1^n(t).$$

Consequently, it follows that

$$\left\| \mathfrak{T}_1^n(t) \right\| \leq \left\| \wp(\varsigma) \left(\lambda_1(t, S_p^{n-1}) - \lambda_1(t, S_p^{n-2}) \right) + \varphi(\varsigma) \int_0^t \left(\lambda_1(\xi, S_p^{n-1}) - \lambda_1(\xi, S_p^{n-2}) \right) d\xi \right\|.$$

Applying Lipchitz's condition,

$$\left\| \mathfrak{T}_1^n(t) \right\| \leq \wp(\varsigma) \chi_1 \left\| S_p - S_p^{n-1} \right\| + \varphi(\varsigma) \chi_1 \left\| S_p - S_p^{n-1} \right\| t.$$

This gives,

$$\left\| \mathfrak{T}_1^n(t) \right\| \leq (\wp(\varsigma) + \varphi(\varsigma)t)^{n+1} \chi_1^{n+1} \zeta.$$

At t_0 , it yields

$$\left\| \mathfrak{T}_1^n(t) \right\| \leq (\wp(\varsigma) + \varphi(\varsigma)t_0)^{n+1} \chi_1^{n+1} \zeta.$$

As n gets closer to ∞ , we obtain

$$\left\| \mathfrak{T}_1^n(t) \right\| \rightarrow 0.$$

Similarly, it can be concluded that

$$\left\| \mathfrak{T}_2^n(t) \right\| \rightarrow 0,$$

$$\left\| \mathfrak{T}_3^n(t) \right\| \rightarrow 0,$$

$$\left\| \mathfrak{T}_4^n(t) \right\| \rightarrow 0.$$

This show that solution existence. It is now necessary to demonstrate that the response is distinct. Therefore, presuming that $S_p(t)$ denotes an alternative method to the suggested model

$$S_p(t) - \hat{S}_p(t) = \wp(\varsigma) (\lambda_1(t, S_p) - \lambda_1(t, \hat{S}_p)) + \varphi(\varsigma) \int_0^t (\lambda_1(\xi, S_p) - \lambda_1(\xi, \hat{S}_p)) d\xi. \quad (19)$$

Equation (19) using the applied norm

$$\left\| S_p(t) - \hat{S}_p(t) \right\| (1 - \wp(\varsigma) \chi_1 + \varphi(\varsigma) \chi_1 t) \leq 0. \quad (20)$$

□

Theorem 3. In this case, the analytical solution of the Caputo fractional model is unique

$$(1 - \wp(\varsigma) \chi_1 + \varphi(\varsigma) \chi_1 t) > 0. \quad (21)$$

Proof. Consider that (21) is equivalent to (20) so that

$$\left\| S_p(t) - \hat{S}_p(t) \right\| (1 - \wp(\varsigma) \chi_1 + \varphi(\varsigma) \chi_1 t) \leq 0.$$

Thus,

$$\left\| S_p(t) - \hat{S}_p(t) \right\| = 0,$$

can be stated. It implies that

$$S_p(t) = \hat{S}_p(t),$$

and that the result is unique. Applying the same process to another function yields the following outcomes.

$$S_i = S_i^1, \quad S_q = S_q^1, \quad S_m = S_m^1, \quad M = M^1.$$

This proof thus demonstrates that, in terms of the Caputo operator, the proposed model (3) has a unique solution. □

Hyers-Ulam-Rassias stability

The Hyers-Ulam-Rassias stability concept is especially used in the study of functional equations. It refers about a functional equation's stability when its arguments are changed slightly. Stated differently, it quantifies the degree to which the solution of a functional equation is similar to the solution of the problem's perturbed version. In many disciplines, including as mathematics, physics, engineering, and economics, the stability of a functional equation is a helpful characteristic for demonstrating the existence and uniqueness of solutions. The stability analysis of the fractional model (3) is covered in this subsection using the Hyers-Ulam-Rassias stability analysis concepts. It is essential to rewrite the model (3) as follows:

$${}^C D_t^\varsigma \vartheta(t) = \lambda(t, \vartheta(t)), \quad (22)$$

$$\vartheta(0) = \vartheta_0, \quad 0 < t < T < \infty,$$

where, the vector $\vartheta = S_p, S_i, S_q, S_m, M$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is a continuous vector function.

Definition 3. Consider the fractional-order is $0 < \varsigma < 1$ and $\lambda : [0, T] \times \mathbf{R}^5 \rightarrow \mathbf{R}^5$ is a continuous mapping. Then, model (22) is Hyers-Ulam stable if $\exists \omega > 0$ and $N > 0$, such that for each solution $\vartheta \in \mathfrak{B}([0, T], \mathbf{R}^5)$, the following inequality is present

$$\left\| {}^C D_t^\varsigma \vartheta - \lambda(t, \vartheta) \right\| \leq N, \quad \forall t \in [0, T],$$

$\exists$ solution $\vartheta' \in \mathfrak{B}([0, T], \mathbf{R}^5)$ of model (22), such as

$$\|\vartheta - \vartheta'\| \leq \omega N, \quad \forall t \in [0, T].$$

Definition 4. Consider that the fractional order ς is $0 < \varsigma < 1$. The function $\lambda : [0, T] \times \mathbf{R}^5 \rightarrow \mathbf{R}^5$ and $\sigma : [0, T] \rightarrow \mathbf{R}^+$ are continuous mappings. Next, model (22) exhibits generalized Hyers-Ulam-Rassias stability with respect to σ if $\exists \mathfrak{B}_{\lambda, \sigma} > 0$, such that for each solution $\vartheta \in \mathfrak{B}([0, T], \mathbf{R}^5)$ the following inequality exists:

$$\left\| {}^C D_t^\varsigma \vartheta(t) - \lambda(t, \vartheta(t)) \right\| \leq \sigma(t), \quad \forall t \in [0, T], \quad (23)$$

$\exists$ solution $\vartheta' \in \mathfrak{B}([0, T], \mathbf{R}^5)$ of model (22), such as

$$\|\vartheta - \vartheta'\| \leq \mathfrak{B}_{\lambda, \sigma} \sigma(t), \quad \forall t \in [0, T].$$

To prove that model (22) is the Hyers-Ulam-Rassias stable, now assume the following

- $[\mathfrak{C}_1] \lambda : [0, T] \times \mathbf{R}^5 \rightarrow \mathbf{R}^5$ is a continuous mapping.
- $[\mathfrak{C}_2] \exists \mathfrak{B}_\lambda > 0$ such that for each solution $\vartheta, \vartheta' \in \mathfrak{B}([0, T], \mathbf{R}^5)$,

$$\|\vartheta - \vartheta'\| \leq \mathfrak{B}_\lambda \|\vartheta - \vartheta'\|, \quad \forall t \in [0, T].$$
- $[\mathfrak{C}_3]$ Let $\sigma \in ([0, T], \mathbf{R}^+)$ be an increasing mapping, and let $\mathfrak{D}_\sigma > 0$, such that

$$\int_0^t \sigma(\xi) d\xi \leq \mathfrak{D}_\sigma \sigma(t), \quad \forall \vartheta \in [0, T].$$

Theorem 4. Consider that $[\mathfrak{C}_1] - [\mathfrak{C}_3]$ exist, and the generalized model (22) Hyers-Ulam-Rassias stability in relation to σ on the interval given that $\wp(\varsigma) \mathfrak{B}_\lambda < 1$.

Proof. Let $\vartheta' \in \mathfrak{B}([0, T], \mathbf{R}^5)$ be a solution of model. Thus, the unique solution of model (22) obtained from

$$\vartheta = \vartheta(0) + \wp(\varsigma) \lambda(t, \vartheta) + \varphi(\varsigma) \int_0^t \lambda(\vartheta, \vartheta(\xi)) d\xi.$$

Based on the facts in (23), it can be concluded that

$$\begin{aligned} & \left\| \vartheta' - \vartheta(0) + \wp(\varsigma) \lambda(t, \vartheta') + \varphi(\varsigma) \int_0^t \lambda(\vartheta, \vartheta'(\xi)) d\xi \right\| \\ & \leq \wp(\varsigma) \sigma(t) + \varphi(\varsigma) \int_0^t \sigma(\xi) d\xi \\ & \leq (\wp(\varsigma) + \varphi(\varsigma) \mathfrak{D}_\sigma) \sigma(t). \end{aligned}$$

So,

$$\begin{aligned} \|\vartheta - \vartheta'\| & \leq \left\| \vartheta' - \vartheta(0) - \wp(\varsigma) \lambda(t, \vartheta') - \varphi(\varsigma) \int_0^t \lambda(\xi, \vartheta'(\xi)) d\xi \right\| \\ & \leq \left\| \vartheta' - \vartheta(0) - \wp(\varsigma) \lambda(t, \vartheta) - \varphi(\varsigma) \int_0^t \lambda(\xi, \vartheta(\xi)) d\xi - \wp(\varsigma) \lambda(t, \vartheta') - \varphi(\varsigma) \int_0^t \lambda(\xi, \vartheta'(\xi)) d\xi \right. \\ & \quad \left. + \wp(\varsigma) \lambda(t, \vartheta') - \varphi(\varsigma) \int_0^t \lambda(\xi, \vartheta'(\xi)) d\xi \right\| \\ & \leq \left\| \vartheta' - \vartheta(0) - \wp(\varsigma) \lambda(t, \vartheta) - \varphi(\varsigma) \int_0^t \lambda(\xi, \vartheta(\xi)) d\xi \right\| + \wp(\varsigma) \|\lambda(t, \vartheta) - \lambda(t, \vartheta')\| \\ & \quad + \varphi(\varsigma) \int_0^t \|\lambda(\xi, \vartheta(\xi)) - \lambda(\xi, \vartheta'(\xi))\| d\xi \\ & \leq (\wp(\varsigma) + \varphi(\varsigma) \mathfrak{D}_\sigma) \sigma(t) + \wp(\varsigma) \mathfrak{B}_\lambda \|\vartheta - \vartheta'\| + \sigma(\varsigma) \mathfrak{B}_\lambda \int_0^t \|\vartheta(\xi) - \vartheta'(\xi)\| d\xi. \end{aligned}$$

Now, $\vartheta(\varsigma)\mathfrak{B}_\lambda < 1$, so

$$\|\vartheta - \vartheta'\| \leq \frac{(\wp(\varsigma) + \varphi(\varsigma)\mathfrak{D}_\sigma)\sigma(t)}{1 - \wp(\varsigma)\mathfrak{B}_\lambda} + \frac{\varphi(\varsigma)\mathfrak{B}_\lambda}{1 - \wp(\varsigma)\mathfrak{B}_\lambda} \int_0^t \|\vartheta(\xi) - \vartheta'(\xi)\| d\xi.$$

Gronwall's inequality results in

$$\|\vartheta - \vartheta'\| \leq \left[\frac{\vartheta(\varsigma) + \varphi(\varsigma)\mathfrak{D}_\sigma}{1 - \wp(\varsigma)\mathfrak{B}_\lambda} \exp(t) \right] \varphi(\varsigma).$$

On setting $\mathfrak{B}_{\lambda,\sigma} = \left[\frac{\wp(\varsigma) + \varphi(\varsigma)\mathfrak{D}_\sigma}{1 - \wp(\varsigma)\mathfrak{B}_\lambda} \exp(t) \right]$. Hence

$$\|\vartheta - \vartheta'\| \leq \mathfrak{B}_{\lambda,\sigma}\sigma(t). \quad (24)$$

The generalized Hyers-Ulam-Rassias stability of model (22) with respect to σ was verified by inequality (24). $\square$

Remark 1. Hyers-Ulam-Rassias stability ensures that small changes in initial conditions or media interventions lead to only minor deviations in smoking behavior predictions. This implies the model is robust and reliable for real-world applications. It supports the design of stable, consistent public health strategies against smoking.

Remark 2. The Lyapunov approach was not used due to challenges in constructing a suitable Lyapunov function for fractional-order systems with memory and non-local effects. Instead, Hyers-Ulam-Rassias stability was adopted as it aligns better with the integral nature of the Caputo derivative. This method provides a direct and effective way to assess solution stability under perturbations.

Chaos control

In this part, the linear feedback control method to stabilize system (3) to equilibrium points. The fractional-order system in its controlled form (3) is analyzed as follows

$$\begin{aligned} {}^C D_t^\varsigma S_p(t) &= \Lambda - \left(\frac{\beta S_p S_i}{N} \right) - \mu S_p + \varepsilon_1 S_m - \varepsilon S_p M + \rho_1 (S_{p_t} - S_p), \\ {}^C D_t^\varsigma S_i(t) &= \left(\frac{\beta S_p S_i}{N} \right) - (\mu + \delta_1 + \gamma) S_i + \nu S_q + \rho_2 (S_{i_t} - S_i), \\ {}^C D_t^\varsigma S_q(t) &= \gamma S_i - (\mu + \delta_2) S_q - \nu S_q + \rho_3 (S_{q_t} - S_q), \\ {}^C D_t^\varsigma S_m(t) &= \varepsilon S_p M - \mu S_m - \varepsilon_1 S_m + \rho_4 (S_{m_t} - S_m) \\ {}^C D_t^\varsigma M(t) &= \phi S_i - m_0 M + \rho_5 (M_t - M), \end{aligned} \quad (25)$$

where $\rho_1, \rho_2, \rho_3, \rho_4$, and ρ_5 are control variables and E^* (3) represents the system equilibrium point. The Jacobian matrix at E^* for system (25) is produced as

$$J(E^*) = \begin{bmatrix} -\mu - \rho_1 & -\beta & 0 & \varepsilon_1 & -\frac{\Lambda}{\mu} \varepsilon \\ 0 & \beta - \gamma - \mu - \rho_2 - \delta_1 & \nu & 0 & 0 \\ 0 & \gamma & -\delta_2 - \mu - \nu - \rho_3 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 - \mu - \rho_4 & \frac{\Lambda}{\mu} \varepsilon \\ 0 & \phi & 0 & 0 & -m_0 - \rho_5 \end{bmatrix},$$

the control parameters $\rho_1 = 1, \rho_2 = 2, \rho_3 = 3, \rho_4 = 4, \rho_5 = 5$ are selected to ensure asymptotic stability by assigning progressively increasing feedback strengths. These values satisfy the conditions for negative eigenvalues in the Jacobian matrix. Given the value of the parameters, the typical roots of the polynomial with respect to the equilibrium point E^0 are as follows.

$$C(\lambda) = \lambda^5 - 278.4299\lambda^4 - 228.1339\lambda^3 - 85.8862\lambda^2 - 15.0862\lambda - 122.0973. \quad (26)$$

Moreover, $\lambda_1 = -1.0001, \lambda_2 = -4.0201, \lambda_3 = -5.0600, \lambda_4 = -1.9900$, and $\lambda_5 = -3.0161$ are computed as the characteristic roots of Equation (26). Since all of the eigenvalues are negative real numbers, E^* is an equilibrium point that is asymptotically stable. Fig. 3 illustrates the chaotic dynamics and phase surface simulations of the smoking model, showcasing the system's behavior under varying conditions. The plots demonstrate how different compartments interact and evolve over time, highlighting the impact of memory effects and media awareness. The chaotic patterns reveal the sensitivity of smoking prevalence to initial conditions and control measures.

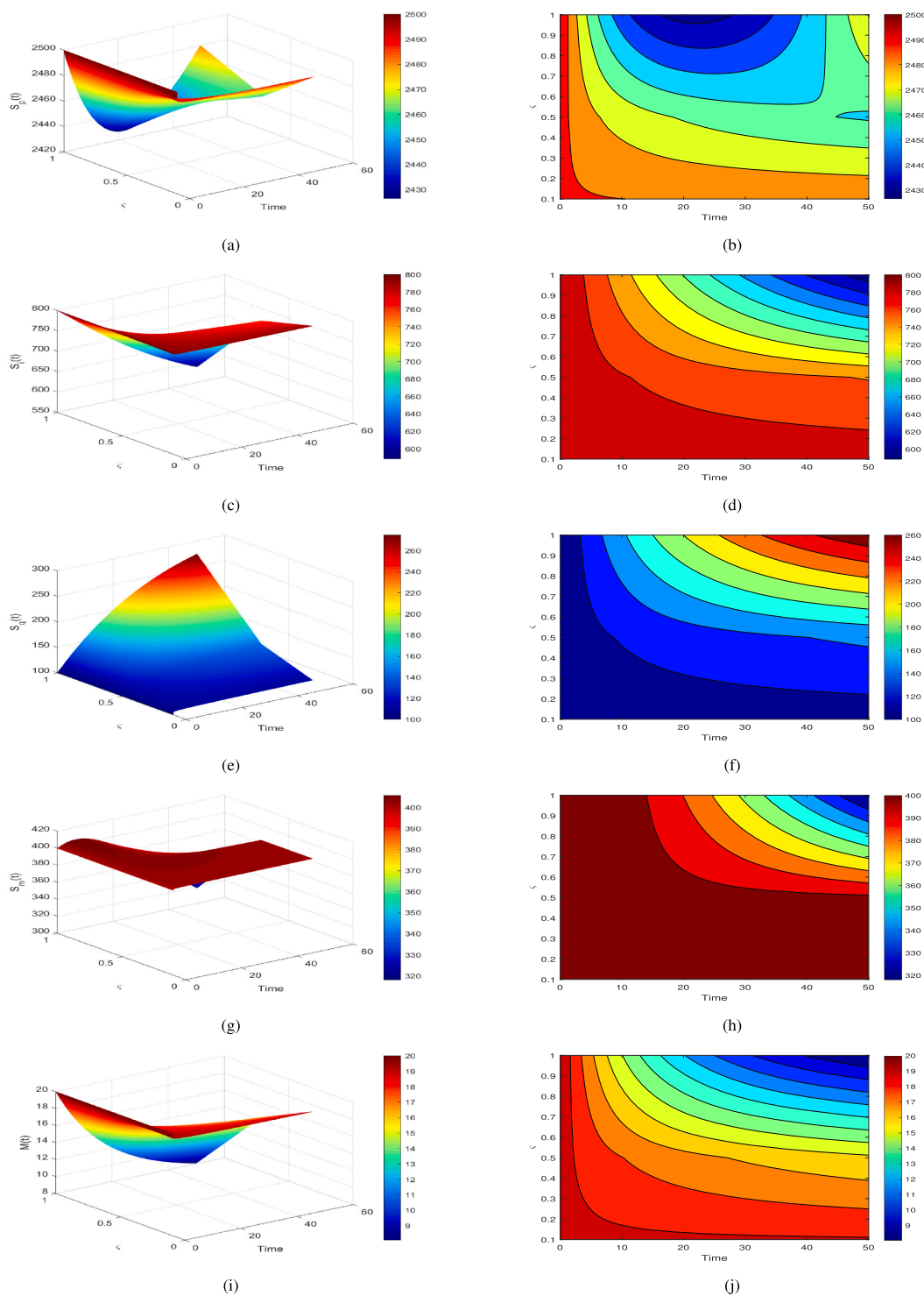


Fig. 3. Chaos dynamics and surface phase simulation of the smoking model.

Method validation

Numerical scheme

In this part, a numerical method is developed for the proposed model (3). To address the numerical solution of the proposed fractional-order model, we developed a novel Newton polynomial-based numerical scheme. This method provides improved accuracy

and efficiency by utilizing higher-order polynomial interpolation tailored for Caputo derivatives. In contrast to traditional schemes, it captures memory effects with reduced computational complexity. Its effectiveness is demonstrated through consistent simulations across varying fractional orders.

$$\begin{cases} {}^C D_t^\varsigma S_p(t) = \Lambda - \left(\frac{\beta S_p S_i}{N} \right) - \mu S_p + \varepsilon_1 S_m - \varepsilon S_p M, \\ {}^C D_t^\varsigma S_i(t) = \left(\frac{\beta S_p S_i}{N} \right) - (\mu + \delta_1 + \gamma) S_i + \nu S_q, \\ {}^C D_t^\varsigma S_q(t) = \gamma S_i - (\mu + \delta_2) S_q - \nu S_q, \\ {}^C D_t^\varsigma S_m(t) = \varepsilon S_p M - \mu S_m - \varepsilon_1 S_m, \\ {}^C D_t^\varsigma M(t) = \phi S_i - m_0 M. \end{cases} \quad (27)$$

For simplicity, the following can be stated regarding the system described above

$$\begin{aligned} {}^C D_t^\varsigma S_p(t) &= \lambda_1(t, S_p), \\ {}^C D_t^\varsigma S_i(t) &= \lambda_2(t, S_i), \\ {}^C D_t^\varsigma S_q(t) &= \lambda_3(t, S_q), \\ {}^C D_t^\varsigma S_m(t) &= \lambda_4(t, S_m), \\ {}^C D_t^\varsigma M(t) &= \lambda_5(t, M). \end{aligned}$$

So,

$$S_p(t) - S_p(0) = \frac{1-\varsigma}{\Gamma(\varsigma)} \int_0^t \lambda_1(\xi, S_p(\xi))(t-\xi)^{\varsigma-1} d\xi.$$

At the point $t_{e+1} = (e+1)\Delta t$, the expression can be written as

$$S_p(t_{e+1}) - S_p(0) = \frac{1-\varsigma}{\Gamma(\varsigma)} \int_0^{t_{e+1}} \lambda_1(\xi, S_p(\xi))(t_{e+1}-\xi)^{\varsigma-1} d\xi.$$

Consequently,

$$S_p(t_{e+1}) = S_p(0) + \frac{1-\varsigma}{\Gamma(\varsigma)} \sum_{v=2}^e \int_{t_v}^{t_{v+1}} \lambda_1(\xi, S_p(\xi))(t_{e+1}-\xi)^{\varsigma-1} d\xi. \quad (28)$$

Equation (28) can be used in place of the Newton polynomial to generate

$$\begin{aligned} S_p^{e+1} = S_{p0} + \frac{1-\varsigma}{\Gamma(\varsigma)} \sum_{v=2}^e \int_{t_v}^{t_{v+1}} & \left\{ \lambda_1(t_{v-2}, S_p^{v-2}) + \frac{\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{\Delta t} (\xi - t_{v-2}) \right. \\ & \left. \frac{\lambda_1(t_v, S_p^v) - 2\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{2(\Delta t)^2} (\xi - t_{v-2})(\xi - t_{v-1}) \right\} \times (t_{e+1} - \xi)^{\varsigma-1} d\xi. \end{aligned} \quad (29)$$

Consequently, the previous equation can be rearranged as follows:

$$\begin{aligned} S_p^{e+1} = S_{p0} + \frac{1-\varsigma}{\Gamma(\varsigma)} \sum_{v=2}^e & \left\{ \int_{t_v}^{t_{v+1}} \lambda_1(t_{v-2}, S_p^{v-2})(t_{e+1}-\xi)^{\varsigma-1} d\xi \right. \\ & + \int_{t_v}^{t_{v+1}} \frac{\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{\Delta t} (\xi - t_{v-2})(t_{e+1}-\xi)^{\varsigma-1} d\xi \\ & \left. + \int_{t_v}^{t_{v+1}} \frac{\lambda_1(t_v, S_p^v) - 2\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{2(\Delta t)^2} (\xi - t_{v-2})(\xi - t_{v-1})(t_{e+1}-\xi)^{\varsigma-1} d\xi \right\}. \end{aligned} \quad (30)$$

Consequently,

$$\begin{aligned}
 S_p^{e+1} &= S_{p0} + \frac{1-\zeta}{\Gamma(\zeta)} \sum_{v=2}^e \lambda_1(t_{v-2}, S_p^{v-2}) \int_{t_v}^{t_{v+1}} (t_{e+1} - \xi)^{\zeta-1} d\xi \\
 &+ \frac{1}{\Gamma(\zeta)} \sum_{v=2}^e \frac{\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{\Delta t} \int_{t_v}^{t_{v+1}} (\xi - t_{v-2})(t_{e+1} - \xi)^{\zeta-1} d\xi \\
 &+ \frac{1}{\Gamma(\zeta)} \sum_{v=2}^e \frac{\lambda_1(t_v, S_p^v) - 2\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})}{2(\Delta t)^2} \int_{t_v}^{t_{v+1}} (\xi - t_{v-2})(\xi - t_{v-1})(t_{e+1} - \xi)^{\zeta-1} d\xi.
 \end{aligned} \quad (31)$$

Equation (31) can be utilized to calculate the integrals mentioned above:

$$\begin{aligned}
 \int_{t_v}^{t_{v+1}} (t_{v+1} - \xi)^{\zeta-1} d\xi &= \frac{(\Delta t)^\zeta}{\zeta} [(e - v + 1)^\zeta - (e - v)^\zeta], \\
 \int_{t_v}^{t_{v+1}} (\xi - t_{v-2})(t_{e+1} - \xi)^{\zeta-1} d\xi &= \frac{(\Delta t)^{\zeta+1}}{\zeta(\zeta+1)} [(e - v + 1)^\zeta(e - v + 3 + 2\zeta) - (e - v)^\zeta(e - v + 3 + 3\zeta)], \\
 \int_{t_v}^{t_{v+1}} (\xi - t_{v-2})(\xi - t_{v-1})(t_{e+1} - \xi)^{\zeta-1} d\xi &= \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \\
 &\times \left[\begin{aligned} &(e - v + 1)^\zeta \left[\begin{aligned} &2(e - v)^2 + (3\zeta + 10)(e - v) \\ &+ 2\zeta^2 + 9\zeta + 12 \end{aligned} \right] \\ &- (e - v)^\zeta \left[\begin{aligned} &2(e - v)^2 + (5\zeta + 10)(e - v) \\ &+ 6\zeta^2 + 18\zeta + 12 \end{aligned} \right] \end{aligned} \right].
 \end{aligned}$$

By substituting these values into Equation (31), the following approach can be observed

$$\begin{aligned}
 S_p^{e+1} &= S_{p0} + \frac{(\Delta t)^\zeta}{\Gamma(\zeta+1)} \sum_{v=2}^e \lambda_1(t_{v-2}, S_p^{v-2}) \times \mathfrak{S} \\
 &+ \frac{(\Delta t)^\zeta}{\Gamma(\zeta+2)} \sum_{v=2}^e [\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})] \times \mathfrak{S}_1 \\
 &+ \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \sum_{v=2}^e [\lambda_1(t_v, S_p^v) - 2\lambda_1(t_{v-1}, S_p^{v-1}) - \lambda_1(t_{v-2}, S_p^{v-2})] \times \mathfrak{S}_2.
 \end{aligned} \quad (32)$$

In a similar manner, the system equations in (27) can be expressed as follows

$$\begin{aligned}
 S_i^{e+1} &= S_{i0} + \frac{(\Delta t)^\zeta}{\Gamma(\zeta+1)} \sum_{v=2}^e \lambda_2(t_{v-2}, S_i^{v-2}) \times \mathfrak{S} \\
 &+ \frac{(\Delta t)^\zeta}{\Gamma(\zeta+2)} \sum_{v=2}^e [\lambda_2(t_{v-1}, S_i^{v-1}) - \lambda_2(t_{v-2}, S_i^{v-2})] \times \mathfrak{S}_1 \\
 &+ \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \sum_{v=2}^e [\lambda_2(t_v, S_i^v) - 2\lambda_2(t_{v-1}, S_i^{v-1}) - \lambda_2(t_{v-2}, S_i^{v-2})] \times \mathfrak{S}_2, \\
 S_q^{e+1} &= S_{q0} + \frac{(\Delta t)^\zeta}{\Gamma(\zeta+1)} \sum_{v=2}^e \lambda_3(t_{v-2}, S_q^{v-2}) \times \mathfrak{S} \\
 &+ \frac{(\Delta t)^\zeta}{\Gamma(\zeta+2)} \sum_{v=2}^e [\lambda_3(t_{v-1}, S_q^{v-1}) - \lambda_3(t_{v-2}, S_q^{v-2})] \times \mathfrak{S}_1 \\
 &+ \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \sum_{v=2}^e [\lambda_3(t_v, S_q^v) - 2\lambda_3(t_{v-1}, S_q^{v-1}) - \lambda_3(t_{v-2}, S_q^{v-2})] \times \mathfrak{S}_2,
 \end{aligned}$$

$$\begin{aligned}
S_m^{e+1} &= S_{m0} + \frac{(\Delta t)^\zeta}{\Gamma(\zeta+1)} \sum_{v=2}^e \lambda_4(t_{v-2}, S_m^{v-2}) \times \mathfrak{S} \\
&+ \frac{(\Delta t)^\zeta}{\Gamma(\zeta+2)} \sum_{v=2}^e [\lambda_4(t_{v-1}, S_m^{v-1}) - \lambda_4(t_{v-2}, S_m^{v-2})] \times \mathfrak{S}_1 \\
&+ \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \sum_{v=2}^e [\lambda_4(t_v, S_m^v) - 2\lambda_4(t_{v-1}, S_m^{v-1}) - \lambda_4(t_{v-2}, S_m^{v-2})] \times \mathfrak{S}_2, \\
M^{e+1} &= M_0 + \frac{(\Delta t)^\zeta}{\Gamma(\zeta+1)} \sum_{v=2}^e \lambda_5(t_{v-2}, M^{v-2}) \times \mathfrak{S} \\
&+ \frac{(\Delta t)^\zeta}{\Gamma(\zeta+2)} \sum_{v=2}^e [\lambda_5(t_{v-1}, M^{v-1}) - \lambda_5(t_{v-2}, M^{v-2})] \times \mathfrak{S}_1 \\
&+ \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \sum_{v=2}^e [\lambda_5(t_v, M^v) - 2\lambda_5(t_{v-1}, M^{v-1}) - \lambda_5(t_{v-2}, M^{v-2})] \times \mathfrak{S}_2.
\end{aligned}$$

Where

$$\begin{aligned}
\mathfrak{S} &= [(e-v+1)^\zeta - (e-v)^\zeta], \\
\mathfrak{S}_1 &= [(e-v+1)^\zeta(e-v+3+2\zeta) - (e-v)^\zeta(e-v+3+3\zeta)], \\
\mathfrak{S}_2 &= \begin{bmatrix} (e-v+1)^\zeta \left[\begin{matrix} 2(e-v)^2 + (3\zeta+10)(e-v) \\ +2\zeta^2 + 9\zeta + 12 \end{matrix} \right] \\ -(e-v)^\zeta \left[\begin{matrix} 2(e-v)^2 + (5\zeta+10)(e-v) \\ +6\zeta^2 + 18\zeta + 12 \end{matrix} \right] \end{bmatrix}.
\end{aligned}$$

Results of proposed scheme

The numerical simulations of the fractional-order model (3) using the Caputo fractional derivative are intended to be shown in this section. Table 1 lists the parameters and initial values in a biologically feasible manner. We apply the numerical solution presented for the Caputo fractional derivative approximation to the model.

The Caputo derivative is used by the model to produce numerical outputs for various fractional values depending on the steady-state point. Approximate solutions against various fractional orders $\zeta = (0.1, 0.95, 0.90, 0.85)$ are plotted in figures (4). Figure (4a), shows that S_p decreases over time for all fractional orders. Lower ζ values (0.95, 0.90, 0.85) lead to a steeper initial decline but a faster stabilization. A lower fractional order suggests that individuals are influenced more by awareness and quitting efforts, reducing the number of potential smokers more rapidly. S_i decreases over time in Fig (4b), indicating that people quit smoking. Decreasing ζ , the decline is faster, meaning smokers quit more efficiently. Memory effects in fractional-order systems imply that past smoking-related influences (health issues, social pressure, media) make quitting more effective. Fig (4c), shows that S_q increases initially as more smokers quit, then stabilizes. Decreasing fractional order, the peak number of quitters is reached faster. Lower fractional orders suggest stronger memory effects, making past anti-smoking campaigns and awareness efforts more impactful. S_m grows slightly in Fig (4d), indicating that awareness spreads over time. Lower fractional order leads to a slower initial increase but stabilizes at a slightly higher level. Awareness builds up gradually, but with stronger memory effects, its impact is more sustained. Media exposure remains nearly constant in Fig (4e). Minor variations indicate that the media's impact is relatively stable across different memory effects. Since media campaigns are ongoing, their influence does not change drastically over time. Decreasing the fractional order (ζ) enhances memory effects, leading to a sharper initial response but faster stabilization. Awareness and quitting behaviors are more effective at lower ζ , suggesting that public health strategies should focus on long-term awareness campaigns. The results confirm that fractional-order models provide a more realistic representation of smoking dynamics compared to classical models.

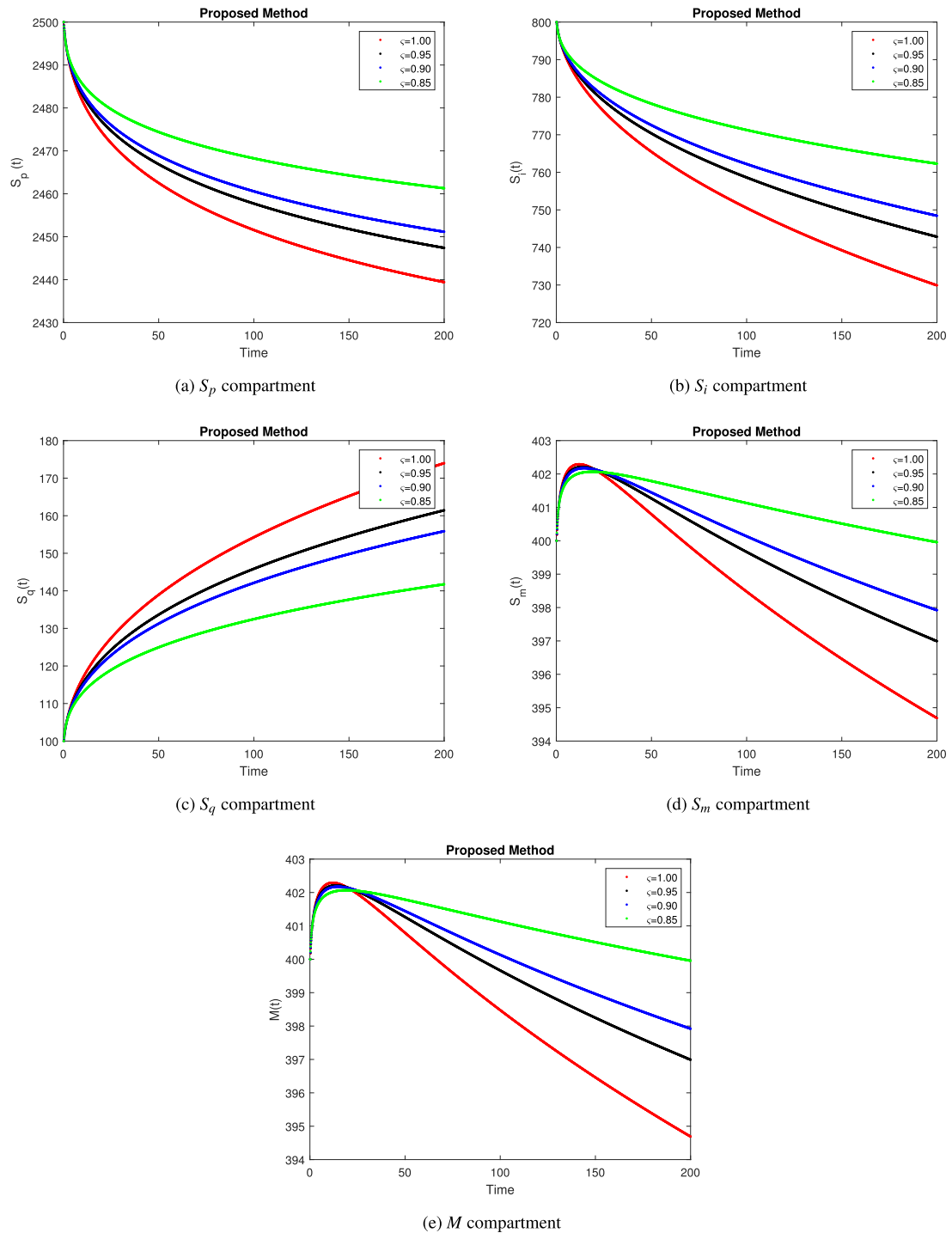


Fig. 4. Graphical representation of the fractional-order smoking model under different values of ζ (a) S_p , (b) S_i , (c) S_q , (d) S_m , and (e) M .

Conclusion

In this study, we developed a robust fractional-order mathematical framework to analyze the dynamics of smoking behavior, employing Caputo derivatives to capture the memory-dependent nature of such behavioral processes. The model successfully addresses limitations inherent in classical integer-order approaches by incorporating long-term historical influences, thus offering a more realistic representation of smoking prevalence and cessation dynamics. The stability of the proposed system was rigorously established

through Hyers-Ulam-Rassias analysis, ensuring reliability under small perturbations. The implementation of chaos control techniques demonstrated the model's capacity to achieve stable long-term behavior, which is crucial for practical public health applications. Furthermore, the Newton polynomial-based numerical scheme proved to be both accurate and computationally efficient for simulating the complex nonlinear dynamics involved. Importantly, the model allows for the integration of delayed effects from past media campaigns and behavioral inertia, thereby providing a valuable decision-support tool for public health planners. The main challenges included handling the nonlocal nature of the Caputo fractional derivative and ensuring the Lipschitz continuity for proving existence and uniqueness. Deriving a suitable iterative scheme for stability analysis also required careful estimation. Additionally, developing an accurate numerical method to preserve memory effects posed significant computational complexity. Future work should focus on enhancing the models scope and applicability. This includes incorporating broader social determinants such as peer influence, economic constraints, and stress-induced smoking behavior. Validation with real-world epidemiological data from diverse demographic groups will enhance its credibility and applicability. The development of multi-scale models that bridge individual-level behavior with population-level outcomes could offer deeper insights into the dynamics of smoking. Furthermore, incorporating machine learning methods may help identify optimal intervention strategies, especially under constrained public health resources. Finally, the modeling framework can be adapted to investigate other addictive behaviors that exhibit memory-dependent patterns. These proposed directions aim to strengthen the models practical relevance while preserving its theoretical integrity, paving the way for more effective and targeted tobacco control strategies globally.

CRediT authorship contribution statement

Muhammad Farman: Writing – Original draft, Validation, Supervision, Resources, and Methodology; **Cicik Alfiniyah:** Methodology, Visualization, Software, Funding acquisition, Conceptualization, Writing – review and Editing; **Khadija Jamil:** Investigation, Formal analysis, Data curation, and Conceptualization; **Aceng Sambas:** Validation, Administrative, Formal analysis; **Nashrul Millah:** Numerical analysis, review and editing; **Ahmadin:** Investigation and Numerical analysis.

Limitations

None

Ethics statements

The data used in this study is fully available within the manuscript and has been obtained from publicly available sources or previously published studies. All data sources have been appropriately cited in the References section. No additional ethical approval was required for this study, as no new data involving human or animal subjects were collected.

Supplementary material and/or additional information [OPTIONAL]

None

Data availability

No data was used for the research described in the article.
remark

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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