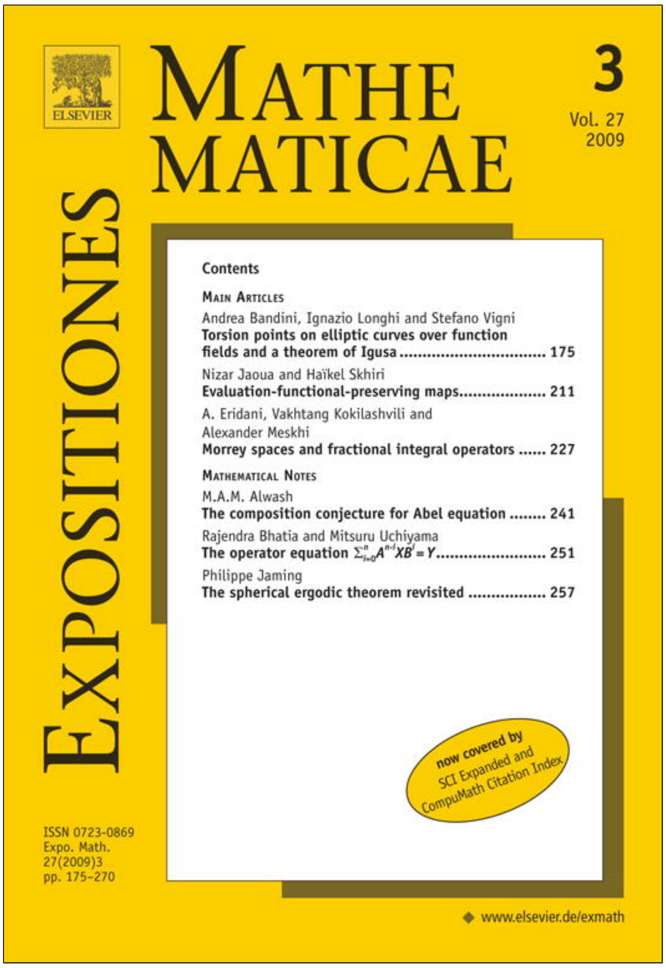

ELSEVIER
MATHEMATICAE
3
Vol. 27
2009
EXPOSITIONES
Contents
Main Articles
Andrea Bandini, Ignazio Longhi and Stefano Vigni
Torsion points on elliptic curves over function fields and a theorem of Igusa 175
Nizar Jaoua and Haïkel Skhiri
Evaluation-functional-preserving maps........ 211
A. Eridani, Vakhtang Kokilashvili and Alexander Meskhi
Morrey spaces and fractional integral operators 227
Mathematical Notes
M.A.M. Alwash
The composition conjecture for Abel equation 241
Rajendra Bhatia and Mitsuru Uchiyama
The operator equation $\sum_{j=0}^{n} A^{n-j}XB^{j} = Y$........ 251
Philippe Jaming
The spherical ergodic theorem revisited 257
now covered by SCI Expanded and CompuMath Citation Index
ISSN 0723-0869
Expo. Math.
27(2009)3
pp. 175–270
www.elsevier.de/exmath

Expositiones Mathematicae

Aims & Scope

Expositiones Mathematicae publishes articles in English, French or German in all branches of mathematics under the headings "Main Articles" and "Mathematical Notes":

Main Articles: These may be (I) expository essays, (II) surveys, (III) original research articles, (IV) historical studies concerning 19th and 20th century mathematics.

Mathematical Notes: These are shorter articles containing new results, new proofs of known theorems or novel points of view.

Our aim is to publish papers of interest to a wide mathematical audience. Our main interest is in expository articles that make high-level research results more widely accessible. In general, material that presents undergraduate-level mathematics is not appropriate for Expositiones Mathematicae.

Main articles must be written in such a way that a graduate-level research student interested in the topic of the paper can read them profitably. When the topic is quite specialized, or the main focus is a narrow research result, the paper is probably not appropriate for this journal.

Mathematical notes can be more focused than main articles. They should address an important mathematical question with reasonably broad appeal. Elementary solutions of elementary problems are typically not appropriate. Neither are overly technical papers, which should best be submitted to a specialized research journal.

Clarity of exposition, accuracy of details and the relevance and interest of the subject matter will be the decisive factors in our acceptance of an article for publication.

All papers are refereed.

Abstracted / Indexed in

CompuMath Citation Index; MathSciNet; ISI Alerting Service; Science Citation Index Expanded; Scopus; Zentralblatt MATH

Search... All fields Search

Help | Advanced Search

Login

Showing 1–49 of 49 results for all: Morrey spaces and fractional integral operat

Searching in archive **math**. Search in all archives.

Morrey spaces and fractional integral operators

All fields

Search

Show abstracts Hide abstracts

Advanced Search

50 results per page. Sort results by Announcement date (newest first) Go

1. arXiv:2007.00468 [pdf, ps, other] math.FA

Commutators of integral operators with functions in Campanato spaces on Orlicz-Morrey spaces

Authors: Minglei Shi, Ryutaro Arai, Eiichi Nakai

Abstract: ...on Orlicz-**Morrey spaces**, where T is a Calderón-Zygmund **operator**, I_ρ is a generalized **fractional integral operator** and b is a function in generalized Cam... ▽ More

Submitted 30 June, 2020; **originally announced** July 2020.

Comments: 45 pages. arXiv admin note: text overlap with arXiv:1812.09148

MSC Class: 42B35; 46E30; 42B20; 42B25

2. arXiv:1911.08573 [pdf, other] math.CA

On optimal parameters involved with two-weighted estimates of commutators of singular and fractional operators with Lipschitz symbols

Authors: Gladis Pradolini, Jorgelina Recchi

Abstract: In this paper we prove two-weighted norm estimates for higher order commutator of singular **integral** and **fractional** type **operators** between weighted L^p and certain... ▽ More

Submitted 18 November, 2019; **originally announced** November 2019.

Comments: 21 pages, 3 figures. arXiv admin note: text overlap with arXiv:1910.10315

MSC Class: 42B20; 42B25; 42B35

3. arXiv:1910.10315 [pdf, other] math.AP

On two weighted problems for commutators of classical operators with optimal behaviour on the parameters involved and extrapolation results

Authors: Gladis Pradolini, Wilfredo Ramos, Jorgelina Recchi

Abstract: We give two weighted norm estimates for higher order commutator of classical **operators** such as singular **integral** and **fractional** type **operators**, between weighted L^p and certain... ▽ More

Submitted 22 October, 2019; **originally announced** October 2019.

4. arXiv:1907.03573 [pdf, ps, other] math.CA

Morrey spaces for Schrödinger operators with nonnegative potentials, fractional integral operators and the Adams inequality on the Heisenberg groups

Authors: Hua Wang

Abstract: ...be a Schrödinger **operator** on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sublaplacian on $\mathbb{H}^n$ and the nonnegative potential V belongs to the reverse Hölder class RH_s with $s \in [Q/2, \infty)$. Here $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. For given $\alpha \in (0, Q)$, the... ▽ More

Submitted 3 July, 2019; **originally announced** July 2019.

Comments: 29 pages. arXiv admin note: substantial text overlap with arXiv:1802.08550

MSC Class: 42B20; 35J10; 22E25; 22E30

5. arXiv:1905.10946 [pdf, ps, other] math.CA

Weighted estimates for bilinear fractional integral operators and their commutators on Morrey spaces

Authors: Qianjun He, Mingquan Wei, Dunyan Yan

Abstract: This paper mainly dedicates to prove a plethora of weighted estimates on **Morrey spaces** for bilinear **fractional integral operators** and their general commutators with BMO functions of the form... ▽ More

Submitted 26 May, 2019; **originally announced** May 2019.

6. arXiv:1905.10082 [pdf, ps, other] math.FA

Bilinear estimates on Morrey spaces by using average

Authors: Naoya Hatano

Abstract: This paper is a follow up of [6]. We investigate the boundedness of the bilinear **fractional integral operator** introduced by Grafakos in [3]. When the local **integrability** index s falls 1 with weights and t exceeds 1, He and Yan obtained... ▽ More

Submitted 24 May, 2019; originally announced May 2019.
Comments: 9 PAGES
MSC Class: 42B35

7. arXiv:1904.00574 [pdf, ps, other] math.FA

A note on the bilinear fractional integral operator acting on Morrey spaces

Authors: Naoya Hatano, Yoshihiro Sawano

Abstract: The boundedness of the bilinear **fractional**... ▽ More

Submitted 1 April, 2019; originally announced April 2019.
Comments: 9 PAGES
MSC Class: 42B35

8. arXiv:1812.03649 [pdf, ps, other] math.FA

Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz--Morrey spaces of the third kind

Authors: Fatih Deringoz, Vagif S. Guliyev, Eiichi Nakai, Yoshihiro Sawano, Minglei Shi

Abstract: In the present paper, we will characterize the boundedness of the generalized **fractional integral operators** I_ρ and the generalized **fractional** maximal **operators** M_ρ on Orlicz... ▽ More

Submitted 10 December, 2018; originally announced December 2018.
MSC Class: 42B20; 42B25; 42B35; 46E30

9. arXiv:1811.06702 [pdf, ps, other] math.AP math.FA

A characterization of rough fractional type integral operators and Campanato estimates for their commutators on the variable exponent vanishing generalized Morrey spaces

Authors: Ferit Grbz, Shenghu Ding, Huili Han, Pinhong Long

Abstract: In this paper, applying some properties of variable exponent analysis, we first dwell on Adams and Spanne type estimates for a class of **fractional** type... ▽ More

Submitted 16 November, 2018; originally announced November 2018.
Comments: 26 pages
MSC Class: 42B20; 42B35; 46E30

10. arXiv:1809.08851 [pdf, ps, other] math.FA

On the behaviors of rough fractional type sublinear operators on vanishing generalized weighted Morrey spaces

Authors: Ferit Gürbüz

Abstract: The aim of this paper is to get the boundedness of rough sublinear **operators** generated by... ▽ More

Submitted 24 September, 2018; originally announced September 2018.

11. arXiv:1808.05189 [pdf, ps, other] math.CA

Some estimates for the bilinear fractional integrals on the Morrey space

Authors: Xiao Yu, Xiangxing Tao, Huihui Zhang, Jianmiao Ruan

Abstract: In this paper, we are interested in the following bilinear **fractional integral operator** $B\mathcal{I}_\alpha$ defined by

B\mathcal{I}_α({f,g})(x)=\int_{% %TCIMACRO{\U{211d} }% %BeginExpansion \mathbb{R} %EndExpansion ^{n}}\frac{f(x-y)g(x+y)}{|y|^{n-α}}dy,

with $0 < \alpha < n$.... ▽ More

Submitted 15 August, 2018; originally announced August 2018.
Comments: 25 pages
MSC Class: 42B20; 42B25 ACM Class: F.2.2

12. arXiv:1806.09293 [pdf, ps, other] math.FA

Mixed Morrey spaces

Authors: Toru Nogayama

Abstract: We introduce mixed **Morrey**... ▽ More

Submitted 25 June, 2018; originally announced June 2018.

13. arXiv:1805.01846 [pdf, ps, other] math.CA

Bilinear fractional integral operators on Morrey spaces

Authors: Qianjun He, Dunyan Yan

Abstract: We prove a plethora of boundedness property of the Adams type for bilinear **fractional integral operators** of the form

$$B_\alpha(f,g)(x)=\int_{\mathbb{R}^n}\frac{f(x-y)g(x+y)}{|y|^{n-\alpha}}dy, \qquad 0<\alpha<n.$$

For $1 < t \le s < \infty$, we prove the non-weighted case through... ▽ More

Submitted 26 May, 2019; v1 submitted 4 May, 2018; originally announced May 2018.

14. arXiv:1804.08718 [pdf, ps, other] math.FA

A characterization for fractional integral and its commutators in Orlicz and generalized Orlicz-Morrey spaces on spaces of homogeneous type

Authors: Vagif S. Guliyev, Fatih Deringoz

Abstract: In this paper, we investigate the boundedness of maximal **operator** and its commutators in generalized Orlicz-… ▽ More
Submitted 23 April, 2018; **originally announced** April 2018.
Comments: 28 pages
MSC Class: 42B20; 42B25; 42B35

15. arXiv:1804.01001 [pdf, ps, other] math.FA

A class of sublinear operators and their commutators by with rough kernels on vanishing generalized Morrey spaces

Authors: Ferit Gurbuz

Abstract: In this paper, we consider the boundedness of a class of sublinear **operators** and their commutators by with rough kernels associated with Calderon-Zygmund…
Submitted 1 April, 2018; **originally announced** April 2018.
Comments: arXiv admin note: text overlap with arXiv:1602.07853, arXiv:1603.03469

16. arXiv:1802.08550 [pdf, ps, other] math.CA

Morrey spaces related to certain nonnegative potentials and fractional integrals on the Heisenberg groups

Authors: Hua Wang

Abstract: …be a Schrödinger **operator** on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on $\mathbb{H}^n$ and the nonnegative potential V belongs to the reverse Hölder class RH_s with $s \geq Q/2$. Here $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. For given $\alpha \in (0, Q)$, the… ▽ More
Submitted 18 February, 2018; **originally announced** February 2018.
Comments: 22 pages. arXiv admin note: text overlap with arXiv:1802.02481
MSC Class: 42B20; 35J10; 22E25; 22E30

17. arXiv:1802.03743 [pdf, ps, other] math.FA

Morrey meets Muckenhoupt: A note on Nakai's generalized Morrey spaces and applications

Authors: Xian Ming Hou, Qingyan Wu, Zunwei Fu, Shanzhen Lu

Abstract: The goal of this paper is to extend Nakai's generalized **Morrey**… ▽ More
Submitted 14 September, 2018; **v1** submitted 11 February, 2018; **originally announced** February 2018.
Comments: 25 pages
MSC Class: 42B35; 42B20; 42B99

18. arXiv:1802.02481 [pdf, ps, other] math.CA

Weighted Morrey spaces related to Schrodinger operators with potentials satisfying a reverse Holder inequality and fractional integrals

Authors: Hua Wang

Abstract: …be a Schrödinger **operator** on $\mathbb{R}^d$, $d \geq 3$, where Δ is the Laplacian **operator** on $\mathbb{R}^d$ and the nonnegative potential V belongs to the reverse Hölder class RH_s for $s \geq d/2$. For given $0 < \alpha < d$, the… ▽ More
Submitted 5 February, 2018; **originally announced** February 2018.
Comments: 30 pages. arXiv admin note: text overlap with arXiv:1801.10217
MSC Class: 42B20; 35J10; 46E30; 47B47

19. arXiv:1801.05275 [pdf, ps, other] math.CA

Two-weight, weak type norm inequalities for fractional integral operators and commutators on weighted Morrey and amalgam spaces

Authors: Hua Wang

Abstract: …be the **fractional integral operator** of order γ, $I_\gamma f(x) = \int_{\mathbb{R}^n} |x-y|^{\gamma-n} f(y)\,dy$, and let $[b, I_\gamma]$ be the linear commutator generated by a symbol function b and I_γ, $[b, I_\gamma]f(x) = b(x) \cdot I_\gamma f(x) - I_\gamma(bf)(x)$. This paper is concerned with two-weight, weak t… ▽ More
Submitted 9 January, 2018; **originally announced** January 2018.
Comments: 32 pages. arXiv admin note: text overlap with arXiv:1712.01269
MSC Class: 42B20; 46E30; 47B38; 47G10

20. arXiv:1705.04050 [pdf, ps, other] math.AP

Norm estimates for Bessel-Riesz operators on generalized Morrey spaces

Authors: Mochammad Idris, Hendra Gunawan, Eridani

Abstract: We revisit the properties of Bessel-Riesz **operators** and refine the proof of the boundedness of these…
Submitted 16 February, 2018; **v1** submitted 11 May, 2017; **originally announced** May 2017.
Comments: 10 pages
MSC Class: 42B20; 26A33; 42B25; 26D10

21. arXiv:1704.05580 [pdf, ps, other] math.PR

Morrey-Campanato estimates for the moments of stochastic integral operators and its application to SPDEs

Authors: Guangying Lv, Hongjun Gao, Jinlong Wei, Jiang-Lun Wu

Abstract: In this paper, we are concerned with the estimates for the moments of stochastic convolution **integrals**. We first deal with the stochastic singular **integral operators** and we aim to derive the **Morrey**-Campanato estimates for the p-moments (f… ▽ More
Submitted 18 April, 2017; **originally announced** April 2017.
Comments: 19 pages
MSC Class: 35K20; 60H15; 60H40

22. arXiv:1703.06395 [pdf, ps, other] math.FA

Sharp estimates for commutators of bilinear operators on Morrey type spaces

Authors: Dinghuai Wang, Jiang Zhou, Zhidong Teng

Abstract: ...the bilinear Calderón-Zygmund **operators** and bilinear **fractional integrals**, respectively. In this paper, it is proved that if $b_1, b_2 \in \mathrm{CMO}$ (the {\rm BMO}-closure of $C_c^\infty(\mathbb{R}^n)$), $[\Pi\vec{b}, T]$ and $[\Pi\vec{b}, I_\alpha]$ ($\vec{b} = (b_1, b_2)$) are al... ▽ More

Submitted 19 March, 2017; **originally announced** March 2017.

Comments: 27 pages. arXiv admin note: text overlap with arXiv:1612.01116

23. arXiv:1702.02411 [pdf, ps, other] math.FA

Multilinear BMO estimates for the commutators of multilinear fractional maximal and integral operators on the product generalized Morrey spaces

Authors: Ferit Gurbuz

Abstract: In this paper, we establish multilinear BMO estimates for commutators of multilinear **fractional** maximal and...

Submitted 8 February, 2017; **originally announced** February 2017.

MSC Class: 42B20; 42B25; 42B35

24. arXiv:1701.07766 [pdf, ps, other] math.FA

Boundedness of Fractional Integral operators and their commutators in vanishing generalized weighted Morrey spaces

Authors: Bilal Çekiç, Ayşegül Çelik Alabalık

Abstract: In this article, we establish some conditions for the boundedness of **fractional integral operators** on the vanishing generalized weighted **Morrey spaces**. We also investigate corresponding commutators ge...

Submitted 16 May, 2017; **v1** submitted 26 January, 2017; **originally announced** January 2017.

Comments: 8 pages

MSC Class: 42B20; 42B35; 46E30

25. arXiv:1701.00850 [pdf, ps, other] math.FA

Hardy-Littlewood, Bessel-Riesz, and fractional integral operators in anisotropic Morrey and Campanato spaces

Authors: Michael Ruzhansky, Durvudkhan Suragan, Nurgissa Yessirkegenov

Abstract: We analyse **Morrey**... ▽ More

Submitted 3 January, 2017; **originally announced** January 2017.

Comments: 29 pages

26. arXiv:1612.00663 [pdf, ps, other] math.FA

Fractional operators on weighted Morrey spaces

Authors: Shohei Nakamura, Yoshihiro Sawano, Hitoshi Tanaka

Abstract: A necessary condition and a sufficient condition for one weight norm inequalities on **Morrey**... ▽ More

Submitted 2 December, 2016; **originally announced** December 2016.

27. arXiv:1610.05449 [pdf, ps, other] math.FA

Fractional type multilinear commutators generated by fractional integral with rough variable kernel and local Campanato functions on generalized vanishing local Morrey spaces

Authors: Ferit Gurbuz

Abstract: In this paper, we consider the boundedness of **fractional** type multilinear commutators generated by...

Submitted 13 December, 2016; **v1** submitted 18 October, 2016; **originally announced** October 2016.

Comments: arXiv admin note: text overlap with arXiv:1602.07853

MSC Class: 42B20; 42B25; 42B35

28. arXiv:1606.02791 [pdf, ps, other] math.FA

A note on the boundedness of discrete commutators on Morrey spaces and their preduals

Authors: Yoshihiro Sawano

Abstract: Dyadic **fractional**...

Submitted 8 June, 2016; **originally announced** June 2016.

Comments: This is: Yoshihiro Sawano A note on the boundedness of discrete commutators on **Morrey spaces** and their preduals. J. Anal. Appl. 11 (2013), no. 1-2, 1-26

MSC Class: 26B33; 41E17 (Primary); 42B25; 42B35 (Secondary)

29. arXiv:1605.08326 [pdf, ps, other] math.AP math.CA math.FA doi 10.2140/apde.2019.12.605

The BMO-Dirichlet problem for elliptic systems in the upper-half space and quantitative characterizations of VMO

Authors: José María Martell, Dorina Mitrea, Irina Mitrea, Marius Mitrea

Abstract: ...is a Carleson measure. We establish a regularity result for the BMO-Dirichlet problem in the upper-half **space**: the nontangential pointwise trace of any given smooth null-solutions of L satisfying the above Carleson measure condition belongs to Sarason's **space** VMO if and only if μ_u satsifies a vanishing Carl... ▽ More

Submitted 24 August, 2018; **v1** submitted 26 May, 2016; **originally announced** May 2016.

MSC Class: Primary: 35B65; 35C15; 35J47; 35J57; 35J67; 42B37. Secondary: 35E99; 42B20; 42B30; 42B35

Journal ref: Analysis & PDE 12 (2019) 605-720

30. arXiv:1603.06739 [pdf, ps, other] math.FA

Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces

Authors: Ferit Gurbuz

Abstract: In this paper, we establish BMO estimates for generalized commutators of rough **fractional** maximal and **integral operators** on generalized weighted **Morrey spaces**, respectively.

Submitted 22 March, 2016; **originally announced** March 2016.

Comments: 16 pages. arXiv admin note: text overlap with arXiv:1603.03469

MSC Class: 42B20; 42B25

31. arXiv:1603.04658 [pdf, ps, other] math.CA

Weighted inequalities for fractional integral operators and linear commutators in the Morrey type spaces

Authors: Hua Wang

Abstract: In this paper, we first introduce some new **Morrey** type… ▽ More

Submitted 9 March, 2016; **originally announced** March 2016.

Comments: 34 pages. arXiv admin note: substantial text overlap with arXiv:1603.03912

MSC Class: 42B20; 42B25; 42B35

32. arXiv:1603.00014 [pdf, ps, other] math.FA

Adams-Spanne type estimates for the commutators of fractional type sublinear operators in generalized Morrey spaces on Heisenberg groups

Authors: Ferit Gurbuz

Abstract: In this paper we give BMO (bounded mean oscillation) **space** estimates for commutators of…

Submitted 14 November, 2016; **v1** submitted 29 February, 2016; **originally announced** March 2016.

MSC Class: 42B25; 42B35; 43A15; 43A80

33. arXiv:1602.08788 [pdf, ps, other] math.AP

Adams-Spanne type estimates for parabolic sublinear operators and their commutators by with rough kernels on parabolic generalized Morrey spaces

Authors: Ferit Gurbuz

Abstract: The aim of this paper is to give Adams-Spanne type estimates for parabolic sublinear **operators** and their commutators by with rough kernels generated by parabolic…

Submitted 19 August, 2017; **v1** submitted 25 February, 2016; **originally announced** February 2016.

MSC Class: 42B20; 42B25; 42B35

34. arXiv:1602.07853 [pdf, ps, other] math.AP

Sublinear operators with rough kernel generated by fractional integrals and commutators on generalized vanishing local Morrey spaces

Authors: Ferit Gurbuz

Abstract: In this paper, we consider the norm inequalities for sublinear **operators** with rough kernel generated by… ▽ More

Submitted 31 August, 2016; **v1** submitted 25 February, 2016; **originally announced** February 2016.

Comments: arXiv admin note: text overlap with arXiv:1603.04088, arXiv:1604.01538, arXiv:1603.03469, arXiv:1602.07468, arXiv:1602.08096, arXiv:1602.08788; text overlap with arXiv:1212.6928, arXiv:1208.4788 by other authors

MSC Class: 42B20; 42B25; 42B35

35. arXiv:1602.07468 [pdf, ps, other] math.AP

Multi-sublinear operators generated by multilinear fractional integral operators and commutators on the product generalized local Morrey spaces

Authors: Ferit Gurbuz

Abstract: The aim of this paper is to get the boundedness of certain multi-sublinear **operators** generated by multilinear… ▽ More

Submitted 10 November, 2016; **v1** submitted 24 February, 2016; **originally announced** February 2016.

Comments: arXiv admin note: substantial text overlap with arXiv:1603.04088; text overlap with arXiv:1212.6928 by other authors

MSC Class: 42B20; 42B25; 42B35

36. arXiv:1410.6327 [pdf, ps, other] math.FA math.CA

B^u_w-function spaces and their interpolation

Authors: Eiichi Nakai, Takuya Sobukawa

Abstract: …-function **spaces** which unify Lebesgue, **Morrey**-Campanato, Lipschitz, B^p, CMO, local **Morrey**-type **spaces**, etc., and investigate the interpolation property of B^u_w-function… ▽ More

Submitted 23 October, 2014; **originally announced** October 2014.

Comments: 43 pages

MSC Class: Primary 42B35; 46B70; Secondary 46E30; 46E35; 42B20; 42B25

37. arXiv:1401.1912 [pdf, ps, other] math.FA

Estimates for multilinear commutators of generalized fractional integral operators on weighted Morrey spaces

Authors: He Sha, Tao Xiangxing

Abstract: …be the **fractional integrals** of L for $0<\alpha<n$. Assume that $\vec{b}=(b_1,b_2,\cdots,b_m)$ is a finite family of locally **integrable** functions, then the multilinear commutators generated by $\vec{b}$ and $L^{-\alpha/2}$ is defined by \begin{equation*} L_{\vec{b}}^{-α/2}f=[b_m,\c… ▽ More

Submitted 9 January, 2014; **originally announced** January 2014.

Comments: 16 pages. arXiv admin note: text overlap with arXiv:1203.4407 by other authors

38. arXiv:1310.2139 [pdf, ps, other] math.CA doi 10.1007/s11118-014-9397-6

Weighted local estimates for fractional type operators

Authors: Alberto Torchinsky

Abstract: …for general **fractional** type **operators** T, where $M^\sharp_{0,s}$ is the local sharp maximal function and M_γ the **fractional** maximal function, as well as a local version of this estimate. This allows us to express the local weighted control of Tf by $M_\gamma f$. Similar estimat… ▽ More

Submitted 8 October, 2013; **originally announced** October 2013.

Comments: arXiv admin note: substantial text overlap with arXiv:1308.1134

MSC Class: 26A33; 31B10

39. arXiv:1305.6684 [pdf, ps, other] math.FA

Weak and strong type estimates for fractional integral operators on Morrey spaces in metric measure spaces

Authors: I. Sihwaningrum, Y. Sawano

Abstract: We discuss here a weak and strong type estimate for **fractional integral operators** on **Morrey spaces**, where the underlying measure μ does not always satisfy the doubling condition.

Submitted 28 May, 2013; **originally announced** May 2013.

Comments: 6 pages

40. arXiv:1303.4480 [pdf, ps, other] math.CA

Multilinear singular and fractional integral operators on weighted Morrey spaces

Authors: Hua Wang, Wentan Yi

Abstract: In this paper, we will study the boundedness properties of multilinear Calderon--Zygmund **operators** and multilinear **fractional integrals** on products of weighted **Morrey spaces** with multiple weights.

Submitted 18 March, 2013; **originally announced** March 2013.

Comments: 21 pages

MSC Class: 42B20; 42B35

41. arXiv:1212.6928 [pdf, ps, other] math.FA math.AP

Generalized local Morrey spaces and fractional integral operators with rough kernel

Authors: Vagif S. Guliyev

Abstract: ...be the **fractional** maximal and **integral operators** with rough kernels, where $0 < \a < n$. In this paper, we shall study the continuity properties of $M_{Ω,\a}$ and $I_{Ω,\a}$ on the generalized local...

Submitted 31 December, 2012; **originally announced** December 2012.

Comments: arXiv admin note: text overlap with arXiv:1203.1441 by other authors

42. arXiv:1203.4407 [pdf, ps, other] math.FA

Commutator Theorems for Fractional Integral Operators on Weighted Morrey Spaces

Authors: Zengyan Si

Abstract: ...be the **fractional integrals** of L for $0 < \alpha < n$. For any locally **integrable** function b, The commutators associated with $L^{-\alpha/2}$ are defined by $[b, L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x)$. When $b \in BMO(\omega)$(weighted BMO... ▽ More

Submitted 21 March, 2012; **v1** submitted 20 March, 2012; **originally announced** March 2012.

Comments: 12 pages; submitted

43. arXiv:1203.4337 [pdf, ps, other] math.FA

Necessary and sufficient conditions for boundedness of commutators of the general fractional integral operators on weighted Morrey spaces

Authors: Zengyan Si, Fayou Zhao

Abstract: ...of the multiplication **operator** by b and the general **fractional integral operator** $L^{-\alpha/2}$ is bounded from the weighed **Morrey space** $L^{p,k}(\omega)$ to... ▽ More

Submitted 20 March, 2012; **originally announced** March 2012.

Comments: 12 pages; Classical Analysis and ODEs (math.CA), Functional Analysis (math.FA)

44. arXiv:1203.1441 [pdf, ps, other] math.CA

Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces

Authors: Hua Wang

Abstract: ...be the **fractional** maximal and **integral operators** with rough kernels, where $0 < \alpha < n$. In this paper, we shall study the continuity properties of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ on the weighted...

Submitted 7 March, 2012; **originally announced** March 2012.

Comments: 12 pages

MSC Class: 42B20; 42B25

45. arXiv:1202.5740 [pdf, ps, other] math.CA

Some estimates for commutators of fractional integrals associated to operators with Gaussian kernel bounds on weighted Morrey spaces

Authors: Hua Wang

Abstract: ...be the **fractional integrals** of L for $0 < \alpha < n$. In this paper, we will obtain some boundedness properties of commutators $[b, L^{-\alpha/2}]$ on the weighted **Morrey spaces** $L^{p,\kappa}(w)$ when the symbol b belongs to... ▽ More

Submitted 26 February, 2012; **originally announced** February 2012.

Comments: 15 pages

MSC Class: 42B20; 42B35

46. arXiv:1111.5463 [pdf, ps, other] math.FA

The Boundedness of Multilinear operators with rough kernel on the weighted Morrey spaces

Authors: He Sha

Abstract: ..., the multilinear **fractional**... ▽ More

Submitted 6 November, 2012; **v1** submitted 23 November, 2011; **originally announced** November 2011.

47. arXiv:1111.5109 [pdf, ps, other] math.FA

Boundedness of oscillatory integral operators and their commutators on weighted Morrey spaces

Authors: Zunwei Fu, Shaoguang Shi, Shanzhen Lu

Abstract: It is proved that both oscillatory **integral**...

Submitted 22 November, 2011; **originally announced** November 2011.

Comments: 15 pages

MSC Class: 42B20 (Primary) 42B25 (Secondary)

48. arXiv:1010.2638 [pdf, ps, other] math.CA

Some estimates for commutators of fractional integral operators on weighted Morrey spaces

Authors: Hua Wang

Abstract: ...be the **fractional integral operator**. In this paper, we shall use a unified approach to show some boundedness properties of commutators $[b, I_\alpha]$ on the weighted **Morrey spaces** $L^{p,\kappa}(w)$ under appro...

Submitted 22 January, 2013; **v1** submitted 13 October, 2010; **originally announced** October 2010.

Comments: 22 pages. arXiv admin note: substantial text overlap with arXiv:1010.2637

MSC Class: 42B20; 42B35

49. arXiv:0806.2391 [pdf, ps, other] math.FA

Morrey Spaces and Fractional Integral Operators

Authors: Eridani, Vakhtang Kokilashvili, Alexander Meskhi

Abstract: The present paper is devoted to the boundedness of **fractional**... ▽ More

Submitted 14 June, 2008; **originally announced** June 2008.

Comments: 13 pages

MSC Class: 42B35; 47B38

Expositiones Mathematicae
Volume 27, Issue 3, 2009, Pages 227-239

Morrey spaces and fractional integral operators

A. Eridani [a, b], Vakhtang Kokilashvili [c], Alexander Meskhi [b, c]

Show more

https://doi.org/10.1016/j.exmath.2009.01.001

Abstract

The present paper is devoted to the boundedness of fractional integral operators in Morrey spaces defined on quasimetric measure spaces. In particular, Sobolev, trace and weighted inequalities with power weights for potential operators are established. In the case when measure satisfies the doubling condition the derived conditions are simultaneously necessary and sufficient for appropriate inequalities.

MSC

primary, 26A33; secondary, 42B35; 47B38

Keywords

Fractional integrals; Morrey spaces; Non-homogeneous spaces; Trace inequality; Two-weight inequality

1. Introduction

The main purpose of this paper is to establish the boundedness of fractional integral operators in (weighted) Morrey spaces defined on quasimetric measure spaces. We derive Sobolev, trace and two-weight inequalities for fractional integrals. In particular, we generalize: (a) Adams [1] trace inequality; (b) the theorem by Stein and Weiss [18] regarding the two-weight inequality for the Riesz potentials; (c) Sobolev-type inequality. We emphasize that in the most cases the derived conditions are necessary and sufficient for appropriate inequalities.

In the paper [9] (see also [10, Chapter 2]) integral-type sufficient condition guaranteeing the two-weight weak-type inequality for integral operator with positive kernel defined on non-homogeneous spaces was established. In the same paper (see also [10, Chapter 2]) the authors solved the two-weight problem for kernel operators on spaces of homogeneous type.

In [12] (see also [5, Chapter 6]) a complete description of non-doubling measure μ guaranteeing the boundedness of fractional integral operator I_α (see the next section for the definition) from $L^p(\mu,X)$ to $L^q(\mu,X)$, $1<p<q<\infty$, was given. We notice that this result was derived in [11] for potentials on Euclidean spaces. In [12], theorems of Sobolev and Adams type for fractional integrals defined on quasimetric measure spaces were established. For the boundedness of fractional integrals on metric measure spaces we refer also to [7]. Some two-weight norm inequalities for fractional operators on R^n with non-doubling measure were studied in [8]. Further, in the paper [13] necessary and sufficient conditions on measure μ governing the inequality of Stein–Weiss type on non-homogeneous spaces were established. For some properties of fractional integrals defined on R^n in weighted Lebesgue spaces with power type weights see e.g., [16, Chapter 5].

The boundedness of the Riesz potential in Morrey spaces defined on Euclidean spaces was studied in [15], [2]. The same problem for fractional integrals on R^n with non-doubling measure was investigated in [17].

Finally, we mention that necessary and sufficient conditions for the boundedness of maximal operators and Riesz potentials in the local Morrey-type spaces were derived in [3], [4].

The main results of this paper were presented in [6].

It should be emphasized that the results of this work are new even for Euclidean spaces.

Constants (often different constants in the same series of inequalities) will generally be denoted by *c* or *C*.

2. Preliminaries

Throughout the paper we assume that $X:=(X,\rho,\mu)$ is a topological space, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a function (quasimetric) $\rho:X\times X\longrightarrow[0,\infty)$ satisfying the

(1) $\rho(x,y)>0$ for all $x\neq y$, and $\rho(x,x)=0$ for all $x\in X$;

(2) there exists a constant $a_0\geqslant 1$, such that $\rho(x,y)\leqslant a_0\rho(y,x)$ for all $x,y\in X$;

(3) there exists a constant $a_1\geqslant 1$, such that $\rho(x,y)\leqslant a_1(\rho(x,z)+\rho(z,y))$ for all $x,y,z\in X$.

We assume that the balls $B(a,r):=\{x\in X:\rho(a,x)<r\}$ are μ-measurable and $0<\mu(B(a,r))<\infty$ for $a\in X, r>0$. For every neighborhood V of $x\in X$, there exists $r>0$, such that $B(x,r)\subset V$. We also assume that $\mu(X)=\infty$, $\mu\{a\}=0$, and $B(a,r_2)\setminus B(a,r_1)\neq\emptyset$, for all $a\in X$, $0<r_1<r_2<\infty$.

The triple (X,ρ,μ) will be called quasimetric measure space.

Let $0<\alpha<1$. We consider the fractional integral operators I_α, and K_α given by

$$I_\alpha f(x):=\int_X f(y)\rho(x,y)^{\alpha-1}d\mu(y),$$

$$K_\alpha f(x):=\int_X f(y)(\mu B(x,\rho(x,y)))^{\alpha-1}d\mu(y),$$

for suitable f on X.

Suppose that ν is another measure on X, $\lambda\geqslant 0$ and $1\leqslant p<\infty$. We deal with the Morrey space $L^{p,\lambda}(X,\nu,\mu)$, which is the set of all functions $f\in L^p_{loc}(X,\nu)$ such that

$$\|f\|_{L^{p,\lambda}(X,\nu,\mu)}:=\sup_B\left(\frac{1}{\mu(B)^\lambda}\int_B|f(y)|^p d\nu(y)\right)^{1/p}<\infty,$$

where the supremum is taken over all balls B.

If $\nu=\mu$, then we have the classical Morrey space $L^{p,\lambda}(X,\mu)$ with measure μ. When $\lambda=0$, then $L^{p,\lambda}(X,\nu,\mu)=L^p(X,\nu)$ is the Lebesgue space with measure ν.

Further, suppose that $\beta\in\mathbb{R}$. We are also interested in weighted Morrey space $M_\beta^{p,\lambda}(X,\mu)$ which is the set of all μ-measurable functions f such that

$$\|f\|_{M_\beta^{p,\lambda}(X,\mu)}:=\sup_{a\in X;r>0}\left(\frac{1}{r^\lambda}\int_{B(a,r)}|f(y)|^p\rho(a,y)^\beta d\mu(y)\right)^{1/p}<\infty.$$

If $\beta=0$, then we denote $M_\beta^{p,\lambda}(X,\mu):=M^{p,\lambda}(X,\mu)$.

We say that a measure μ satisfies the growth condition ($\mu\in(GC)$), if there exists $C_0>0$ such that $\mu(B(a,r))\leqslant C_0 r$; further, μ satisfies the doubling condition ($\mu\in(DC)$) if $\mu(B(a,2r))\leqslant C_1\mu(B(a,r))$ for some $C_1>1$. If $\mu\in(DC)$, then (X,ρ,μ) is called a space of homogeneous type (SHT). A quasimetric measure space (X,ρ,μ), where the doubling condition is not assumed, is also called a non-homogeneous space.

The measure μ on X satisfies the reverse doubling condition ($\mu\in(RDC)$) if there are constants η_1 and η_2 with $\eta_1>1$ and $\eta_2>1$ such that

$$\mu B(x,\eta_1 r)\geqslant\eta_2\mu B(x,r). \qquad (1)$$

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Regular/SuppMathOperators.js

It is known (see e.g., [19, p. 11]) that if $\mu\in(DC)$, then $\mu\in(RDC)$.

The next statements are from [12] (see also [5, Theorem 6.1.1, Corollary 6.1.1] and [11] in the case of Euclidean spaces).

Theorem A

Let (X,ρ,μ) *be a quasimetric measure space. Suppose that* $1<p<q<\infty$ *and* $0<\alpha<1$. *Then* I_α *is bounded from* $L^p(X)$ *to* $L^q(X)$ *if and only if there exists a positive constant C such that*

$$\mu(B(a,r))\leqslant Cr^s,\quad s=\frac{pq(1-\alpha)}{pq+p-q}, \tag{2}$$

for all $a\in X$ *and* $r>0$.

Corollary B

Let (X,ρ,μ) *be a quasimetric measure space,* $1<p<1/\alpha$ *and* $1/q=1/p-\alpha$. *Then* I_α *is bounded from* $L^p(X)$ *to* $L^q(X)$ *if and only if* $\mu\in(GC)$.

The latter statement by different proof was also derived in [7] for metric spaces.

To prove some of our statements we need the following Hardy-type transform:

$$H_a f(x):=\int_{\rho(a,y)\leqslant\rho(a,x)} f(y)\,d\mu(y),$$

where *a* is a fixed point of *X* and $f\in L_{loc}(X,\mu)$.

Theorem C

Suppose that (X,ρ,μ) *is a quasimetric measure space and* $1<p\leqslant q<\infty$. *Assume that* ν *is another measures on X. Let* V (*resp.* W) *be non-negative* $\nu\times\nu$*-measurable* (*resp. non-negative* $\mu\times\mu$*-measurable*) *function on* $X\times X$. *If there exists a positive constant C independent of* $a\in X$ *and* $t>0$ *such that*

$$\left(\int_{\rho(a,y)\geqslant t} V(a,y)\,d\nu(y)\right)^{1/q}\left(\int_{\rho(a,y)\leqslant t} W(a,y)^{1-p'}\,d\mu(y)\right)^{1/p'}\leqslant C<\infty,$$

then there exists a positive constant c such that for all μ*-measurable non-negative f and* $a\in X$ *the inequality*

$$\left(\int_{B(a,r)}(H_a f(x))^q V(a,x)\,d\nu(x)\right)^{1/q}\leqslant c\left(\int_{B(a,r)}(f(x))^p W(a,x)\,d\mu(x)\right)^{1/p}$$

holds.

This statement was proved in [5, Section 1.1] for Lebesgue spaces.

Proof of Theorem C

Let $f\geqslant 0$. We define $S(s):=\int_{\rho(a,y)<s} f(y)\,d\mu(y)$, for $s\in[0,r]$. Suppose $S(r)<\infty$, then $2^m<S(r)\leqslant 2^{m+1}$, for some $m\in\mathbb{Z}$. Let

$$s_j:=\sup\{t:S(t)\leqslant 2^j\},\ j\leqslant m \ \text{and}\ s_{m+1}:=r.$$

Then it is easy to see that (see also [5, pp. 5–8] for details) $(s_j)_{j=-\infty}^{m+1}$ is a non-decreasing sequence, $S(s_j)\leqslant 2^j$, $S(t)\geqslant 2^j$ for $t>s_j$ and

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Regular/SuppMathOperators.js

$$2^j\leqslant\int_{s_j\leqslant\rho(a,y)\leqslant s_{j+1}}f(y)\,d\mu(y).$$

If $\beta:=\lim_{j\to-\infty}s_j$, then

$$\rho(a,x)<r\Leftrightarrow\rho(a,x)\in[0,\beta]\cup\bigcup_{j=-\infty}^{m}(s_j,s_{j+1}].$$

If $S(r)=\infty$, then we may put $m=\infty$. Since

$$0\leqslant\int_{\rho(a,y)<\beta}f(y)\,d\mu(y)\leqslant S(s_j)\leqslant 2^j,$$

for every j, therefore $\int_{\rho(a,y)<\beta}f(y)\,d\mu(y)=0$. From these observations, we have

$$\int_{\rho(a,x)<r}(H_af(x))^qV(a,x)\,d\nu(x)\leqslant\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}}(H_af(x))^qV(a,x)\,d\nu(x)\leqslant\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}}V(a,x)\left(\int_{\rho(a,y)\leqslant s_{j+1}}(f(y))\,d\mu(y)\right)^q d\nu(x).$$

Notice that

$$\int_{\rho(a,y)\leqslant s_{j+1}}f\,d\mu\leqslant S(s_{j+2})\leqslant 2^{j+2}\leqslant C\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}f\,d\mu.$$

Using Hölder's inequality, we find that

$$\begin{aligned}\int_{\rho(a,x)<r}(H_af(x))^qV(a,x)\,d\mu(x)&\leqslant\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}}V(a,x)\left(\int_{\rho(a,y)\leqslant s_{j+1}}(f(y))\,d\mu(y)\right)^q d\nu(x)\\&\leqslant C\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}}V(a,x)\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}(f(y))\,d\mu(y)\right)^q d\nu(x)\\&\leqslant C\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}}V(a,x)\,d\nu(x)\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}(f(y))^pW(a,y)\,d\mu(y)\right)^{q/p}\times\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\\&\leqslant C\sum_{j=-\infty}^{m}\int_{s_j\leqslant\rho(a,y)}V(a,y)\,d\nu(y)\left(\int_{\rho(a,y)\leqslant s_j}W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}(f(y))^pW(a,y)\,d\mu(y)\right)^{q/p}\\&\leqslant C\sum_{j=-\infty}^{m}\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j}(f(y))^pW(a,y)\,d\mu(y)\right)^{q/p}\leqslant C\left(\int_{\rho(a,y)<r}(f(y))^pW(a,y)\,d\mu(y)\right)^{q/p}.\end{aligned}$$

This completes the proof of the theorem. □

For our purposes we also need the following lemma (see [14] for the case of R^n).

Lemma D

Suppose that (X,ρ,μ) *be an SHT. Let* $0<\lambda<1\leqslant p<\infty$. *Then there exists a positive constant C such that for all balls* B_0,

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)}\leqslant C\mu(B_0)^{(1-\lambda)/p}.$$

Proof

Let $B_0:=B(x_0,r_0)$ and $B:=B(a,r)$. We have

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)}=\sup_B\left(\frac{\mu(B_0\cap B)}{\mu(B)^{\lambda}}\right)^{1/p}.$$

Suppose that $B_0\cap B\neq\emptyset$. Let us assume that $r\leqslant r_0$. Then (see [19, Lemma 1] or [10, p. 9]) $B\subset B(x_0,br_0)$, where $b=a_1(1+a_0)$. By the doubling condition it follows that

$$\frac{\mu(B\cap B_0)}{\mu(B)^{\lambda}}\leqslant\frac{\mu(B)}{\mu(B)^{\lambda}}=\mu(B)^{1-\lambda}\leqslant\mu(B(x_0,br_0))^{1-\lambda}\leqslant C\mu(B_0)^{1-\lambda}.$$

 Then

$$\frac{\mu(B\cap B_0)}{\mu(B)^{\lambda}}\leqslant c\frac{\mu(B_0)}{\mu(B_0)^{\lambda}}=c\mu(B_0)^{1-\lambda}.\square$$

The next lemma may be well known but we prove it for the completeness.

Lemma E

Let (X,ρ,μ) *be a non-homogeneous space with the growth condition. Suppose that* $\sigma>-1$. *Then there exists a positive constant c such that for all* $a\in X$ *and* $r>0$, *the inequality*

$$I(a,r,\sigma):=\int_{B(a,r)}\rho(a,x)^{\sigma}d\mu\leqslant cr^{\sigma+1}$$

holds.

Proof

Let $\sigma\geqslant 0$. Then the result is obvious because of the growth condition for μ. Further, assume that $-1<\sigma<0$. We have

$$I(a,r,\sigma)=\int_0^{\infty}\mu\{x\in B(a,r):\rho(a,x)^{\sigma}>\lambda\}d\lambda=\int_0^{\infty}\mu(B(a,r)\cap B(a,\lambda^{1/\sigma}))d\lambda=\int_0^{r^{\sigma}}+\int_{r^{\sigma}}^{\infty}:=I^{(1)}(a,r,\sigma)+I^{(2)}(a,r,\sigma).$$

By the growth condition for μ we have

$$I^{(1)}(a,r,\sigma)\leqslant r^{\sigma}\mu(B(a,r))\leqslant cr^{\sigma+1},$$

while for $I^{(2)}(a,r,\sigma)$ we find that

$$I^{(2)}(a,r,\sigma)\leqslant c\int_{r^{\sigma}}^{\infty}\lambda^{1/\sigma}d\lambda=-\frac{c}{(\sigma+1)}\sigma r^{\sigma+1}=c_1r^{\sigma+1}$$

because $1/\sigma<-1$. □

The following statement is the trace inequality for the operator K_{α} (see [1] for the case of Euclidean spaces and, e.g., [10] or [5, Theorem 6.2.1] for an SHT).

Theorem F

Let (X,ρ,μ) *be an SHT. Suppose that* $1<p<q<\infty$ *and* $0<\alpha<1/p$. *Assume that* ν *is another measure on X. Then* K_{α} *is bounded from* $L^p(X,\mu)$ *to* $L^q(X,\nu)$ *if and only if*

$$\nu B\leqslant c(\mu B)^{q(1/p-\alpha)},$$

for all balls B in X.

3. Main results

In this section we formulate the main results of the paper. We begin with the case of an SHT.

Theorem 3.1

Let (X,ρ,μ) *be an SHT and let* $1<p<q<\infty$. *Suppose that* $0<\alpha<1/p$, $0<\lambda_1<1-\alpha p$ *and* $\lambda_2/q=\lambda_1/p$. *Then* K_{α} *is bounded from* $L^{p,\lambda_1}(X,\mu)$ *to* $L^{q,\lambda_2}(X,\nu,\mu)$ *if and only if there is a positive constant c such that*

 (3)

for all balls B.

The next statement is a consequence of Theorem 3.1.

Theorem 3.2

Let (X,ρ,μ) *be an SHT and let* $1<p<q<\infty$. *Suppose that* $0<\alpha<1/p$, $0<\lambda_1<1-\alpha p$ *and* $\lambda_2/q=\lambda_1/p$. *Then for the boundedness of* K_α *from* $L^{p,\lambda_1}(X,\mu)$ *to* $L^{q,\lambda_2}(X,\mu)$ *it is necessary and sufficient that* $q=p/(1-\alpha p)$.

For non-homogeneous spaces we have the following statements:

Theorem 3.3

Let (X,ρ,μ) *be a non-homogeneous space with the growth condition. Suppose that* $1<p\leqslant q<\infty$, $1/p-1/q\leqslant\alpha<1$ *and* $\alpha\neq 1/p$. *Suppose also that* $p\alpha-1<\beta<p-1$, $0<\lambda_1<\beta-\alpha p+1$ *and* $\lambda_1 q=\lambda_2 p$. *Then* I_α *is bounded from* $M_\beta^{p,\lambda_1}(X,\mu)$ *to* $M_\gamma^{q,\lambda_2}(X,\mu)$, *where* $\gamma=q(1/p+\beta/p-\alpha)-1$.

Theorem 3.4

Suppose that (X,ρ,μ) *is a quasimetric measure space and* μ *satisfies condition* (2). *Let* $1<p<q<\infty$. *Assume that* $0<\alpha<1$, $0<\lambda_1<p/q$ *and* $s\lambda_1/p=\lambda_2/q$. *Then the operator* I_α *is bounded from* $M^{p,\lambda_1 s}(X,\mu)$ *to* $M^{q,\lambda_2}(X,\mu)$.

4. Proof of the main results

In this section we give the proofs of the main results.

Proof of Theorem 3.1

Necessity: Suppose K_α is bounded from $L^{p,\lambda_1}(\mu)$ to $L^{q,\lambda_2}(X,\nu,\mu)$. Fix $B_0:=B(x_0,r_0)$. For $x,y\in B_0$, we have that

$$B(x,\rho(x,y))\subseteq B(x,a_1(a_0+1)r_0)\subseteq B(x_0,a_1(1+a_1(a_0+1))r_0).$$

Hence using the doubling condition for μ, it is easy to see that

$$\mu(B_0)^\alpha\leqslant cK_\alpha\chi_{B_0}(x),\quad x\in B_0.$$

Consequently, using the condition $\lambda_2/q=\lambda_1/p$, the boundedness of K_α from $L^{p,\lambda}(X,\mu)$ to $L^{q,\lambda_2}(X,\nu,\mu)$ and Lemma D we find that

$$\mu(B_0)^{\alpha-\lambda_1/p}\nu(B_0)^{1/q}\leqslant c\|K_\alpha\chi_{B_0}\|_{L^{q,\lambda_2}(X,\nu,\mu)}\leqslant c\|\chi_{B_0}\|_{L^{p,\lambda_1}(X,\mu)}\leqslant c\mu(B_0)^{(1-\lambda_1)/p}.$$

Since *c* does not depend on B_0 we have condition (3).

Sufficiency: Let $B:=B(a,r)$, $\tilde{B}:=B(a,2a_1r)$ and $f\geqslant 0$. Write $f\in L^{p,\lambda_1}(\mu)$ as $f=f_1+f_2:=f\chi_{\tilde{B}}+f\chi_{\tilde{B}^C}$, where χ_B is a characteristic function of *B*. Then we have

$$S:=\int_B(K_\alpha f(x))^q d\nu(x)\leqslant c\int_B(K_\alpha f_1(x))^q d\nu(x)+\int_B(K_\alpha f_2(x))^q d\nu(x):=c(S_1+S_2).$$

$S1\leqslant\int X(K\alpha f1)q(x)d\nu(x)\leqslant c\int B(a,2a1r)(f(x))pd\mu(x)q/p.$

Now observe that if $\rho(a,x)<r$ and $\rho(a,y)>2a1r$, then $\rho(a,y)>2a1\rho(a,x)$. Consequently, using the facts $\mu\in(RDC)$ (see (1)), $0<\lambda1<1-\alpha p$ and condition (3) we have

$S2\leqslant c\int B(a,r)\int\rho(a,y)>rf(y)\mu B(a,\rho(a,y))1-\alpha d\mu(y)qd\nu(x)=\nu(B)\sum k=0\infty\int B(a,\eta1k+1r)\setminus B(a,\eta1kr)f(y)\mu B(a,\rho(a,y))1-\alpha d\mu(y)q\leqslant c\nu(B)\sum k=0\infty\int B(a,\eta1k+1r)(f(y))pd\mu(y)1/p\times\int B(a,\eta1k+1r)\setminus B(a,\eta1kr)\mu B(a,\rho(a,y))(\alpha-1)p'd\mu(y)1/p'q\leqslant c\|f\|Lp,\lambda1(X,\mu)q\nu(B)\sum k=0\infty\mu B(a,\eta1k+1r)\lambda1/p+\alpha-1+1/p'q\leqslant c\|f\|Lp,\lambda1(X,\mu)q\nu(B)\mu(B)(\lambda1/p+\alpha-1/p)q\sum k=0\infty\eta2k(\lambda1/p+\alpha-1/p)q\leqslant c\|f\|Lp,\lambda1(X,\mu)q\mu(B)q\lambda1/p=c\|f\|Lp,\lambda1(X,\mu)q\mu(B)\lambda2,$

where the positive constant *c* does not depend on *B*. Now the result follows immediately. □

Proof of Theorem 3.2

Sufficiency: Assuming $\alpha=1/p-1/q$ and $\mu=\nu$ in Theorem 3.1 we have that $K\alpha$ is bounded from $Lp,\lambda1(X,\mu)$ to $Lq,\lambda2(X,\mu)$.

Necessity: Suppose that $K\alpha$ is bounded from $Lp,\lambda1(X,\mu)$ to $Lq,\lambda2(X,\mu)$. Then by Theorem 3.1 we have

$\mu(B)1/q-1/p+\alpha\leqslant c.$

The conditions $\mu(X)=\infty$ and $\mu\{x\}=0$, for all $x\in X$, implies that $\alpha=1/p-1/q$. □

Proof of Theorem 3.3

Let $f\geqslant0$. For $x,a\in X$, let us introduce the following notation:

$E1(x):=y:\rho(a,y)\rho(a,x)<12a1,E2(x):=y:12a1\leqslant\rho(a,y)\rho(a,x)\leqslant2a1,E3(x):=y:2a1<\rho(a,y)\rho(a,x).$

For $i=1,2,3$, $r>0$ and $a\in X$, we denote

$Si:=\int\rho(a,x)<r\rho(a,x)\gamma\int Ei(x)f(y)\rho(x,y)\alpha-1d\mu(y)qd\mu(x).$

If $y\in E1(x)$, then $\rho(a,x)<2a1a0\rho(x,y)$. Hence, it is easy to see that

$S1\leqslant C\int B\rho(a,x)\gamma+q(\alpha-1)\int\rho(a,y)<\rho(a,x)f(y)d\mu(y)qd\mu(x).$

Taking into account the condition $\gamma<(1-\alpha)q-1$ we have

$\int\rho(a,x)>t\rho(a,x)\gamma+q(\alpha-1)d\mu(x)=\sum n=0\infty\int B(a,2k+1t)\setminus B(a,2kt)(\rho(a,x))\gamma+(\alpha-1)qd\mu(x)\leqslant c\sum n=0\infty(2kt)\gamma+q(\alpha-1)+1=ct\gamma+q(\alpha-1)+1,$

while the condition $\beta<p-1$ implies

$\int\rho(a,x)<t\rho(a,x)\beta(1-p')+1d\mu(x)\leqslant ct\beta(1-p')+1.$

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Regular/SuppMathOperators.js

$$\sup_{a\in X, t>0}\left(\int_{\rho(a,x)>t}\rho(a,x)^{\gamma+q(\alpha-1)}d\mu(x)\right)^{1/q}\left(\int_{B(a,t)}\rho(a,y)^{\beta(1-p')}d\mu(y)\right)^{1/p'}<\infty.$$

Now using Theorem C we have

$$S_1\leqslant c\left(\int_B\rho(a,x)^{\beta}(f(y))d\mu(y)\right)^{q/p}\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_1 q/p}=c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_2}.$$

Further, observe that if $\rho(a,y)>2a_1\rho(a,x)$, then $\rho(a,y)\leqslant a_1\rho(a,x)+a_1\rho(a,y)\leqslant\rho(a,y)/2+a_1\rho(x,y)$. Hence $\rho(a,y)/(2a_1)\leqslant\rho(x,y)$. Consequently, using the growth condition for μ, the fact $\lambda_1<\beta-\alpha p+1$ and Lemma E we find that

$$S_3\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\int_{\rho(a,y)>\rho(a,x)}f(y)\rho(a,y)^{1-\alpha}d\mu(y)\right)^q d\mu(x)\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\sum_{k=0}^{\infty}\int_{B(a,2^{k+1}\rho(a,x))\setminus B(a,2^k\rho(a,x))}f(y)\rho(a,y)^{1-\alpha}d\mu(y)\right)^q d\mu(x)\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\sum_{k=0}^{\infty}\left(\int_{B(a,2^{k+1}\rho(a,x))}f^p(y)\rho(a,y)^{\beta}d\mu(y)\right)^{1/p}\times\left(\int_{B(a,2^{k+1}\rho(a,x))\setminus B(a,2^k\rho(a,x))}\rho(a\right.\right.$$
$$\left.\left.p')+(\alpha-1)p'}d\mu(y)\right)^{1/p'}\right)^q d\mu(x)\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\gamma}\times\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1-\beta/p}(\mu B(a,2^{k+1}\rho(a,x)))^{1/p'}\right)^q d\mu(x)\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1/p-\beta/p}\right)^q d\mu(x)\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{(\lambda_1/p+\alpha-1/p-\beta/p)q+\gamma}d\mu(x)=c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\lambda_1 q/p-1}d\mu(x)\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_1 q/p}=c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_2}.$$

So, we conclude that

$$S_3\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_2}.$$

To estimate S_2 we consider two cases. First assume that $\alpha<1/p$. Let

$$E_{k,r}:=\{x:2^k r\leqslant\rho(a,x)<2^{k+1}r\},\quad F_{k,r}:=\{x:2^{k-1}r/a_1\leqslant\rho(a,x)<a_1 2^{k+2}r\}.$$

Assume that $p^*=p/(1-\alpha p)$. By Hölder's inequality, Corollary B and the assumption $\gamma=q(1/p+\beta/p-\alpha)-1$ we have

$$S_2=\sum_{k=-\infty}^{-1}\int_{E_{k,r}}\rho(a,x)^{\gamma}\left(\int_{E_2(x)}f(y)\rho(x,y)^{\alpha-1}d\mu(y)\right)^q d\mu(x)\leqslant\sum_{k=-\infty}^{-1}\left(\int_{E_{k,r}}\rho(a,x)^{\gamma}\left(\int_{E_2(x)}f(y)\rho(x,y)^{\alpha-1}d\mu(y)\right)^{p^*}d\mu(x)\right)^{q/p^*}\times\left(\int_{E_{k,r}}\rho(a,x)^{\gamma p^*/(p^*-q)}d\mu(x)\right)^{(p^*-q)/p^*}\leqslant c\sum_{k=-\infty}^{-1}2^{k(\gamma+(p^*-q)/p^*)}\left(\int_X I_{\alpha}(f\chi_{F_{k,r}})(x)^{p^*}d\mu(x)\right)^{q/p^*}\leqslant c\sum_{k=-\infty}^{-1}2^{k(\gamma+(p^*-q)/p^*)}\left(\int_{F_{k,r}}(f(x))^p d\mu(x)\right)^{q/p}\leqslant c\left(\int_{B(a,2a_1r)}\rho(a,x)^{\beta}(f(x))^p d\mu(x)\right)^{q/p}\leqslant c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}r^{\lambda_1 q/p}=c\|f\|^q_{M^{\beta}_{p,\lambda_1}(X,\mu)}q$$

Let us now consider the case $1/p<\alpha<1$.

First notice that (see [13])

$$\int_{E_2(x)}(\rho(x,y))^{(\alpha-1)p'}d\mu(y)\leqslant c\rho(a,x)^{1+(\alpha-1)p'},$$

where the positive constant *c* does not depend on *a* and *x*.

This estimate and Hölder's inequality yield

$$S_2\leqslant c\sum_{k=-\infty}^{-1}\int_{E_{k,r}}\rho(a,x)^{\gamma+[(\alpha-1)p'+1]q/p'}\left(\int_{E_2(x)}(f(y))^p d\mu(y)\right)^{q/p}d\mu(x)^{q/p'}\leqslant c\sum_{k=-\infty}^{-1}\int_{E_{k,r}}\rho(a,x)^{\gamma+[(\alpha-1)p'+1]q/p'}d\mu(x)\left(\int_{F_{k,r}}(f(y))^p d\mu(y)\right)^{q/p}\leqslant c\sum_{k=-\infty}^{-1}(2^k r)^{\gamma+[(\alpha-1)p'+1]q/p'}$$

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Regular/SuppMathOperators.js

+1∫Fk,r(f(y))pdμ(y)q/p=c∑k=-∞-12kβq/p∫Fk,r(f(y))pdμ(y)q/p⩽c∫B(a,2a1r)
(f(y))pρ(a,y)βdμ(y)q/p⩽c‖f‖Mβp,λ1(X,μ)qrλ1q/p=c‖f‖Mβp,λ1(X,μ)qrλ2.

Now the result follows immediately. □

Proof of Theorem 3.4

Let f⩾0. Suppose that a∈X and r>0. Suppose also that f1=fχB(a,2a1r) and f2=f-f1. Then Iαf=Iαf1+Iαf2. Consequently,

∫B(a,r)(Iαf(x))qdμ(x)⩽2q-1∫B(a,r)(Iαf1(x))qdμ(x)+∫B(a,r)(Iαf2(x))qdμ(x):=2q-1(Sa,r(1)+Sa,r(2)).

Due to Theorem A and the condition sλ1/p=λ2/q we have

Sa,r(1)⩽c∫B(a,2a1r)(f(x))pdμ(x)q/p=c1(2a1r)λ1s∫B(a,2a1r)(f(x))pdxq/prλ1sq/p⩽c‖f‖Mp,λ1s(X,μ)qrλ2.

Now observe that if x∈B(a,r) and y∈X∖B(a,2a1r), then ρ(a,y)/2a1⩽ρ(x,y). Hence Hölder's inequality, condition (2) and the condition 0<λ1<p/q yield

Iαf2(x)=∫X∖B(a,2a1r)f(y)/ρ(x,y)1-αdμ(y)=∑k=0∞∫B(a,2k+2a1r)∖B(a,2k+1a1r)
(f(y))pdμ(y)1/p×∫B(a,2k+2a1r)∖B(a,2k+1a1r)ρ(a,y)
(α-1)p′dμ(y)1/p′⩽c∑k=0∞1(2k+1a1r)λ1s∫B(a,2k+1a1r)
(f(y))pdμ(y)1/p(2ka1r)λ1s/p+α-1+s/p′⩽c‖f‖Mp,λ1s(X,μ)rλ1s/p+α-1+s/p′.

Consequently, by the assumptions sλ1/p=λ2/q and s=pq(1-α)/(pq+p-q) we conclude that

Sa,r(2)⩽c‖f‖Mp,λ1s(X,μ)qr(λ1s/p+α-1+s/p′)q+s=c‖f‖Mp,λ1s(X,μ)qrλ2.

Summarizing the estimates derived above we finally have the desired result. □

Acknowledgements

The second and third authors were partially supported by the INTAS Grant no. 05-1000008-8157 and the Georgian National Science Foundation Grant no. GNSF/ST07/3-169.

The authors express their gratitude to the referee and Professor Christian Berg for their remarks and suggestions.

References

[1] D.R. Adams
A trace inequality for generalized potentials
Studia Math., 48 (1973), pp. 99-105
CrossRef View Record in Scopus Google Scholar

[2] D.R. Adams
A note on Riesz potentials
Duke Math. J., 42 (4) (1975), pp. 765-778
CrossRef View Record in Scopus Google Scholar

[3] V. Burenkov, H.V. Guliyev
Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces
Studia Math., 163 (2) (2004), pp. 157-176
CrossRef View Record in Scopus Google Scholar

[4] I.V. Burenkov, H.V. Guliyev, V.S. Guliyev
Necessary and sufficient conditions for the boundedness of the fractional maximal operator in local Morrey-type spaces
Dokl. Akad. Nauk, 409 (4) (2006), pp. 443-447
(in Russian)
Google Scholar

[5] D. Edmunds, V. Kokilashvili, A. Meskhi
Bounded and Compact Integral Operators, Mathematics and its Applications, vol. 543, Kluwer Academic Publishers, Dordrecht, Boston, London (2002)

[6] Eridani, V. Kokilashvili, A. Meskhi, Morrey spaces and fractional integral operators, Preprint No. 65, School of Mathematical Sciences, GC University, Lahore, 2007.
Google Scholar

[7] J. García-Cuerva, A.E. Gatto
Boundedness properties of fractional integral operators associated to non-doubling measures
Studia Math., 162 (3) (2004), pp. 245-261
CrossRef View Record in Scopus Google Scholar

[8] J. García-Cuerva, J.M. Martell
Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces
Indiana Univ. Math. J., 50 (3) (2001), pp. 1241-1280
View Record in Scopus Google Scholar

[9] I. Genebashvili, A. Gogatishvili, V. Kokilashvili
Solution of two-weight problems for integral transforms with positive kernels
Georgian Math. J., 3 (1) (1996), pp. 319-342
View Record in Scopus Google Scholar

[10] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, M. Krbec
Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Regular/SuppMathOperators.js pe, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 92, Longman, Harlow

(1998)

[11] V. Kokilashvili, Weighted estimates for classical integral operators, in: Nonlinear Analysis, Function Spaces and Applications IV, Roudnice nad Labem, vol. 119, Teubner-Texte Math, Leipzig, 1990, pp. 86–103.
Google Scholar

[12] V. Kokilashvili, A. Meskhi
Fractional integrals on measure spaces
Frac. Calc. Appl. Anal., 4 (4) (2001), pp. 1-24
View Record in Scopus Google Scholar

[13] V. Kokilashvili, A. Meskhi
On some weighted inequalities for fractional integrals on non-homogeneous spaces
Z. Anal. Anwend., 24 (4) (2005), pp. 871-885
CrossRef View Record in Scopus Google Scholar

[14] Y. Komori, T. Mizuhara
Notes on commutators and Morrey spaces
Hokkaido Math. J., 32 (2003), pp. 345-353
CrossRef View Record in Scopus Google Scholar

[15] J. Peetre
On the theory of Lp,λ spaces
J. Funct. Anal., 4 (1969), pp. 71-87
Article Download PDF View Record in Scopus Google Scholar

[16] S. Samko, A.A. Kilbas, O.I. Marichev
Fractional Integrals and Derivatives. Theory and Applications
Gordon and Breach, London (1993)
Google Scholar

[17] Y. Sawano, H. Tanaka
Morrey spaces for nondoubling measures
Acta Math. Sinica, 21 (6) (2005), pp. 1535-1544
CrossRef View Record in Scopus Google Scholar

[18] E.M. Stein, G. Weiss
Fractional integrals on *n*-dimensional Euclidean spaces
J. Math. Mech., 7 (4) (1958), pp. 503-514
View Record in Scopus Google Scholar

[19] J.O. Strömberg, A. Torchinsky
Weighted Hardy Spaces, Lecture Notes in Mathematics, vol. 1381, Springer, Berlin (1989)

View Abstract

Available online at www.sciencedirect.com
ScienceDirect
Expo. Math. 27 (2009) 227 – 239
EXPOSITIONES MATHEMATICAE
www.elsevier.de/exmath

Morrey spaces and fractional integral operators

A. Eridani[a,b], Vakhtang Kokilashvili[c], Alexander Meskhi[b,c,*]

[a] *Department of Mathematics, Airlangga University, Campus C, Mulyorejo, Surabaya 60115, Indonesia*
[b] *School of Mathematical Sciences, Government College University, 68-B New Muslim Town, Lahore, Pakistan*
[c] *A. Razmadze Mathematical Institute, M. Aleksidze St., Tbilisi 0193, Georgia*

Received 4 July 2008; received in revised form 8 December 2008

Abstract

The present paper is devoted to the boundedness of fractional integral operators in Morrey spaces defined on quasimetric measure spaces. In particular, Sobolev, trace and weighted inequalities with power weights for potential operators are established. In the case when measure satisfies the doubling condition the derived conditions are simultaneously necessary and sufficient for appropriate inequalities.

MSC 2000: primary 26A33; secondary 42B35; 47B38

Keywords: Fractional integrals; Morrey spaces; Non-homogeneous spaces; Trace inequality; Two-weight inequality

1. Introduction

The main purpose of this paper is to establish the boundedness of fractional integral operators in (weighted) Morrey spaces defined on quasimetric measure spaces. We derive Sobolev, trace and two-weight inequalities for fractional integrals. In particular, we

* Corresponding author at: School of Mathematical Sciences, Government College University, 68-B New Muslim Town, Lahore, Pakistan. Tel.: +995 32 32 62 47; fax: +995 32 36 40 86.

E-mail addresses: keumala_ikhsanti@yahoo.com (A. Eridani), kokil@rmi.acnet.ge (V. Kokilashvili), meskhi@rmi.acnet.ge (A. Meskhi).

0723-0869/$ - see front matter
doi:10.1016/j.exmath.2009.01.001

generalize: (a) Adams [1] trace inequality; (b) the theorem by Stein and Weiss [18] regarding the two-weight inequality for the Riesz potentials; (c) Sobolev-type inequality. We emphasize that in the most cases the derived conditions are necessary and sufficient for appropriate inequalities.

In the paper [9] (see also [10, Chapter 2]) integral-type sufficient condition guaranteeing the two-weight weak-type inequality for integral operator with positive kernel defined on non-homogeneous spaces was established. In the same paper (see also [10, Chapter 2]) the authors solved the two-weight problem for kernel operators on spaces of homogeneous type.

In [12] (see also [5, Chapter 6]) a complete description of non-doubling measure μ guaranteeing the boundedness of fractional integral operator I_α (see the next section for the definition) from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1 < p < q < \infty$, was given. We notice that this result was derived in [11] for potentials on Euclidean spaces. In [12], theorems of Sobolev and Adams type for fractional integrals defined on quasimetric measure spaces were established. For the boundedness of fractional integrals on metric measure spaces we refer also to [7]. Some two-weight norm inequalities for fractional operators on $\mathbf{R}^n$ with non-doubling measure were studied in [8]. Further, in the paper [13] necessary and sufficient conditions on measure μ governing the inequality of Stein–Weiss type on non-homogeneous spaces were established. For some properties of fractional integrals defined on $\mathbf{R}^n$ in weighted Lebesgue spaces with power type weights see e.g., [16, Chapter 5].

The boundedness of the Riesz potential in Morrey spaces defined on Euclidean spaces was studied in [15,2]. The same problem for fractional integrals on $\mathbb{R}^n$ with non-doubling measure was investigated in [17].

Finally, we mention that necessary and sufficient conditions for the boundedness of maximal operators and Riesz potentials in the local Morrey-type spaces were derived in [3,4].

The main results of this paper were presented in [6].

It should be emphasized that the results of this work are new even for Euclidean spaces.

Constants (often different constants in the same series of inequalities) will generally be denoted by c or C.

2. Preliminaries

Throughout the paper we assume that $X := (X, \rho, \mu)$ is a topological space, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (quasimetric) $\rho : X \times X \longrightarrow [0, \infty)$ satisfying the conditions:

(1) $\rho(x, y) > 0$ for all $x \neq y$, and $\rho(x, x) = 0$ for all $x \in X$;
(2) there exists a constant $a_0 \geqslant 1$, such that $\rho(x, y) \leqslant a_0 \rho(y, x)$ for all $x, y \in X$;
(3) there exists a constant $a_1 \geqslant 1$, such that $\rho(x, y) \leqslant a_1(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$.

We assume that the balls $B(a, r) := \{x \in X : \rho(a, x) < r\}$ are μ-measurable and $0 < \mu(B(a, r)) < \infty$ for $a \in X$, $r > 0$. For every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty$, $\mu\{a\} = 0$, and $B(a, r_2) \setminus B(a, r_1) \neq \emptyset$, for all $a \in X$, $0 < r_1 < r_2 < \infty$.

The triple (X, ρ, μ) will be called quasimetric measure space.

Let $0 < \alpha < 1$. We consider the fractional integral operators I_α, and K_α given by

$$I_\alpha f(x) := \int_X f(y)\rho(x, y)^{\alpha-1}\, d\mu(y),$$

$$K_\alpha f(x) := \int_X f(y)(\mu B(x, \rho(x, y)))^{\alpha-1}\, d\mu(y),$$

for suitable f on X.

Suppose that v is another measure on X, $\lambda \geqslant 0$ and $1 \leqslant p < \infty$. We deal with the Morrey space $L^{p,\lambda}(X, v, \mu)$, which is the set of all functions $f \in L^p_{\mathrm{loc}}(X, v)$ such that

$$\|f\|_{L^{p,\lambda}(X,v,\mu)} := \sup_B \left(\frac{1}{\mu(B)^\lambda} \int_B |f(y)|^p\, dv(y) \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B.

If $v=\mu$, then we have the classical Morrey space $L^{p,\lambda}(X, \mu)$ with measure μ. When $\lambda=0$, then $L^{p,\lambda}(X, v, \mu) = L^p(X, v)$ is the Lebesgue space with measure v.

Further, suppose that $\beta \in \mathbf{R}$. We are also interested in weighted Morrey space $M^{p,\lambda}_\beta(X, \mu)$ which is the set of all μ-measurable functions f such that

$$\|f\|_{M^{p,\lambda}_\beta(X,\mu)} := \sup_{a\in X; r>0} \left(\frac{1}{r^\lambda} \int_{B(a,r)} |f(y)|^p \rho(a, y)^\beta\, d\mu(y) \right)^{1/p} < \infty.$$

If $\beta = 0$, then we denote $M^{p,\lambda}_\beta(X, \mu) := M^{p,\lambda}(X, \mu)$.

We say that a measure μ satisfies the growth condition ($\mu \in (GC)$), if there exists $C_0 > 0$ such that $\mu(B(a, r)) \leqslant C_0 r$; further, μ satisfies the doubling condition ($\mu \in (DC)$) if $\mu(B(a, 2r)) \leqslant C_1\, \mu(B(a, r))$ for some $C_1 > 1$. If $\mu \in (DC)$, then (X, ρ, μ) is called a space of homogeneous type (SHT). A quasimetric measure space (X, ρ, μ), where the doubling condition is not assumed, is also called a non-homogeneous space.

The measure μ on X satisfies the reverse doubling condition ($\mu \in (RDC)$) if there are constants η_1 and η_2 with $\eta_1 > 1$ and $\eta_2 > 1$ such that

$$\mu B(x, \eta_1 r) \geqslant \eta_2 \mu B(x, r). \tag{1}$$

It is known (see e.g., [19, p. 11]) that if $\mu \in (DC)$, then $\mu \in (RDC)$.

The next statements are from [12] (see also [5, Theorem 6.1.1, Corollary 6.1.1] and [11] in the case of Euclidean spaces).

Theorem A. *Let (X, ρ, μ) be a quasimetric measure space. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1$. Then I_α is bounded from $L^p(X)$ to $L^q(X)$ if and only if there exists a positive constant C such that*

$$\mu(B(a, r)) \leqslant C r^s, \quad s = \frac{pq(1-\alpha)}{pq + p - q}, \tag{2}$$

for all $a \in X$ and $r > 0$.

Corollary B. *Let (X, ρ, μ) be a quasimetric measure space, $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Then I_α is bounded from $L^p(X)$ to $L^q(X)$ if and only if $\mu \in (GC)$.*

The latter statement by different proof was also derived in [7] for metric spaces.

To prove some of our statements we need the following Hardy-type transform:

$$H_a f(x) := \int_{\rho(a,y) \leqslant \rho(a,x)} f(y)\, d\mu(y),$$

where a is a fixed point of X and $f \in L_{\mathrm{loc}}(X, \mu)$.

Theorem C. *Suppose that (X, ρ, μ) is a quasimetric measure space and $1 < p \leqslant q < \infty$. Assume that ν is another measures on X. Let V (resp. W) be non-negative $\nu \times \nu$-measurable (resp. non-negative $\mu \times \mu$-measurable) function on $X \times X$. If there exists a positive constant C independent of $a \in X$ and $t > 0$ such that*

$$\left(\int_{\rho(a,y) \geqslant t} V(a, y)\, d\nu(y) \right)^{1/q} \left(\int_{\rho(a,y) \leqslant t} W(a, y)^{1-p'}\, d\mu(y) \right)^{1/p'} \leqslant C < \infty,$$

then there exists a positive constant c such that for all μ-measurable non-negative f and $a \in X$ the inequality

$$\left(\int_{B(a,r)} (H_a f(x))^q V(a, x)\, d\nu(x) \right)^{1/q} \leqslant c \left(\int_{B(a,r)} (f(x))^p W(a, x)\, d\mu(x) \right)^{1/p}$$

holds.

This statement was proved in [5, Section 1.1] for Lebesgue spaces.

Proof of Theorem C. Let $f \geqslant 0$. We define $S(s) := \int_{\rho(a,y)<s} f(y)\, d\mu(y)$, for $s \in [0, r]$. Suppose $S(r) < \infty$, then $2^m < S(r) \leqslant 2^{m+1}$, for some $m \in \mathbb{Z}$. Let

$$s_j := \sup\{t : S(t) \leqslant 2^j\}, \quad j \leqslant m \quad \text{and} \quad s_{m+1} := r.$$

Then it is easy to see that (see also [5, pp. 5–8] for details) $(s_j)_{j=-\infty}^{m+1}$ is a non-decreasing sequence, $S(s_j) \leqslant 2^j$, $S(t) \geqslant 2^j$ for $t > s_j$, and

$$2^j \leqslant \int_{s_j \leqslant \rho(a,y) \leqslant s_{j+1}} f(y)\, d\mu(y).$$

If $\beta := \lim_{j \to -\infty} s_j$, then

$$\rho(a, x) < r \quad \Leftrightarrow \quad \rho(a, x) \in [0, \beta] \cup \bigcup_{j=-\infty}^{m} (s_j, s_{j+1}].$$

If $S(r) = \infty$, then we may put $m = \infty$. Since

$$0 \leqslant \int_{\rho(a,y)<\beta} f(y)\, d\mu(y) \leqslant S(s_j) \leqslant 2^j,$$

for every j, therefore $\int_{\rho(a,y)<\beta} f(y)\,d\mu(y) = 0$. From these observations, we have

$$\begin{aligned}
&\int_{\rho(a,x)<r} (H_a f(x))^q V(a,x)\,dv(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,x) \leqslant s_{j+1}} (H_a f(x))^q V(a,x)\,dv(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,x) \leqslant s_{j+1}} V(a,x)\left(\int_{\rho(a,y) \leqslant s_{j+1}} (f(y))\,d\mu(y)\right)^q dv(x).
\end{aligned}$$

Notice that

$$\int_{\rho(a,y) \leqslant s_{j+1}} f\,d\mu \leqslant S(s_{j+2}) \leqslant 2^{j+2} \leqslant C \int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} f\,d\mu.$$

Using Hölder's inequality, we find that

$$\begin{aligned}
&\int_{\rho(a,x)<r} (H_a f(x))^q V(a,x)\,d\mu(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,x) \leqslant s_{j+1}} V(a,x)\left(\int_{\rho(a,y) \leqslant s_{j+1}} (f(y))\,d\mu(y)\right)^q dv(x)\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,x) \leqslant s_{j+1}} V(a,x)\left(\int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} (f(y))\,d\mu(y)\right)^q dv(x)\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,x) \leqslant s_{j+1}} V(a,x)\,dv(x)\\
&\quad \times \left(\int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\quad \times \left(\int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j \leqslant \rho(a,y)} V(a,y)\,dv(y)\left(\int_{\rho(a,y) \leqslant s_j} W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\\
&\left(\int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\leqslant C \sum_{j=-\infty}^{m} \left(\int_{s_{j-1} \leqslant \rho(a,y) \leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\leqslant C \left(\int_{\rho(a,y) \leqslant r} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}.
\end{aligned}$$

This completes the proof of the theorem. □

For our purposes we also need the following lemma (see [14] for the case of $\mathbf{R}^n$).

Lemma D. *Suppose that* (X, ρ, μ) *be an SHT. Let* $0 < \lambda < 1 \leqslant p < \infty$. *Then there exists a positive constant* C *such that for all balls* B_0,

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)} \leqslant C\mu(B_0)^{(1-\lambda)/p}.$$

Proof. Let $B_0 := B(x_0, r_0)$ and $B := B(a, r)$. We have

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)} = \sup_B \left(\frac{\mu(B_0 \cap B)}{\mu(B)^\lambda} \right)^{1/p}.$$

Suppose that $B_0 \cap B \neq \emptyset$. Let us assume that $r \leqslant r_0$. Then (see [19, Lemma 1] or [10, p. 9]) $B \subset B(x_0, br_0)$, where $b = a_1(1 + a_0)$. By the doubling condition it follows that

$$\frac{\mu(B \cap B_0)}{\mu(B)^\lambda} \leqslant \frac{\mu(B)}{\mu(B)^\lambda} = \mu(B)^{1-\lambda} \leqslant \mu(B(x_0, br_0))^{1-\lambda} \leqslant C\mu(B_0)^{1-\lambda}.$$

Let now $r_0 < r$. Then $\mu B_0 \leqslant c\mu B$, where the constant c depends only on a_1 and a_0. Then

$$\frac{\mu(B \cap B_0)}{\mu(B)^\lambda} \leqslant c \frac{\mu(B_0)}{\mu(B_0)^\lambda} = c\mu(B_0)^{1-\lambda}. \qquad \square$$

The next lemma may be well known but we prove it for the completeness.

Lemma E. *Let* (X, ρ, μ) *be a non-homogeneous space with the growth condition. Suppose that* $\sigma > -1$. *Then there exists a positive constant* c *such that for all* $a \in X$ *and* $r > 0$, *the inequality*

$$I(a, r, \sigma) := \int_{B(a,r)} \rho(a, x)^\sigma \, d\mu \leqslant cr^{\sigma+1}$$

holds.

Proof. Let $\sigma \geqslant 0$. Then the result is obvious because of the growth condition for μ. Further, assume that $-1 < \sigma < 0$. We have

$$I(a, r, \sigma) = \int_0^\infty \mu\{x \in B(a, r) : \rho(a, x)^\sigma > \lambda\} \, d\lambda$$
$$= \int_0^\infty \mu(B(a, r) \cap B(a, \lambda^{1/\sigma})) \, d\lambda = \int_0^{r^\sigma} + \int_{r^\sigma}^\infty := I^{(1)}(a, r, \sigma) + I^{(2)}(a, r, \sigma).$$

By the growth condition for μ we have

$$I^{(1)}(a, r, \sigma) \leqslant r^\sigma \mu(B(a, r)) \leqslant cr^{\sigma+1},$$

while for $I^{(2)}(a, r, \sigma)$ we find that

$$I^{(2)}(a, r, \sigma) \leqslant c \int_{r^\sigma}^\infty \lambda^{1/\sigma} \, d\lambda = \frac{-c(\sigma + 1)}{\sigma} r^{\sigma+1} = c_1 r^{\sigma+1}$$

because $1/\sigma < -1$. $\square$

The following statement is the trace inequality for the operator K_α (see [1] for the case of Euclidean spaces and, e.g., [10] or [5, Theorem 6.2.1] for an SHT).

Theorem F. *Let (X, ρ, μ) be an SHT. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Assume that ν is another measure on X. Then K_α is bounded from $L^p(X, \mu)$ to $L^q(X, \nu)$ if and only if*

$$\nu B \leqslant c(\mu B)^{q(1/p-\alpha)},$$

for all balls B in X.

3. Main results

In this section we formulate the main results of the paper. We begin with the case of an SHT.

Theorem 3.1. *Let (X, ρ, μ) be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \lambda_1 < 1 - \alpha p$ and $\lambda_2/q = \lambda_1/p$. Then K_α is bounded from $L^{p,\lambda_1}(X, \mu)$ to $L^{q,\lambda_2}(X, \nu, \mu)$ if and only if there is a positive constant c such that*

$$\nu(B) \leqslant c\mu(B)^{q(1/p-\alpha)}, \tag{3}$$

for all balls B.

The next statement is a consequence of Theorem 3.1.

Theorem 3.2. *Let (X, ρ, μ) be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \lambda_1 < 1 - \alpha p$ and $\lambda_2/q = \lambda_1/p$. Then for the boundedness of K_α from $L^{p,\lambda_1}(X, \mu)$ to $L^{q,\lambda_2}(X, \mu)$ it is necessary and sufficient that $q = p/(1 - \alpha p)$.*

For non-homogeneous spaces we have the following statements:

Theorem 3.3. *Let (X, ρ, μ) be a non-homogeneous space with the growth condition. Suppose that $1 < p \leqslant q < \infty$, $1/p - 1/q \leqslant \alpha < 1$ and $\alpha \neq 1/p$. Suppose also that $p\alpha - 1 < \beta < p - 1$, $0 < \lambda_1 < \beta - \alpha p + 1$ and $\lambda_1 q = \lambda_2 p$. Then I_α is bounded from $M_\beta^{p,\lambda_1}(X, \mu)$ to $M_\gamma^{q,\lambda_2}(X, \mu)$, where $\gamma = q(1/p + \beta/p - \alpha) - 1$.*

Theorem 3.4. *Suppose that (X, ρ, μ) is a quasimetric measure space and μ satisfies condition (2). Let $1 < p < q < \infty$. Assume that $0 < \alpha < 1$, $0 < \lambda_1 < p/q$ and $s\lambda_1/p = \lambda_2/q$. Then the operator I_α is bounded from $M^{p,\lambda_1 s}(X, \mu)$ to $M^{q,\lambda_2}(X, \mu)$.*

4. Proof of the main results

In this section we give the proofs of the main results.

Proof of Theorem 3.1. *Necessity*: Suppose K_α is bounded from $L^{p,\lambda_1}(\mu)$ to $L^{q,\lambda_2}(X, \nu, \mu)$. Fix $B_0 := B(x_0, r_0)$. For $x, y \in B_0$, we have that

$$B(x, \rho(x, y)) \subseteq B(x, a_1(a_0 + 1)r_0) \subseteq B(x_0, a_1(1 + a_1(a_0 + 1))r_0).$$

Hence using the doubling condition for μ, it is easy to see that

$$\mu(B_0)^{\alpha} \leqslant c K_{\alpha}\chi_{B_0}(x), \quad x \in B_0.$$

Consequently, using the condition $\lambda_2/q = \lambda_1/p$, the boundedness of K_α from $L^{p,\lambda}(X,\mu)$ to $L^{q,\lambda_2}(X,\nu,\mu)$ and Lemma D we find that

$$\mu(B_0)^{\alpha-\lambda_1/p}\nu(B_0)^{1/q} \leqslant c\|K_\alpha\chi_{B_0}\|_{L^{q,\lambda_2}(X,\nu,\mu)} \leqslant c\|\chi_{B_0}\|_{L^{p,\lambda_1}(X,\mu)} \leqslant c\mu(B_0)^{(1-\lambda_1)/p}.$$

Since c does not depend on B_0 we have condition (3).

Sufficiency: Let $B := B(a,r)$, $\tilde{B} := B(a, 2a_1 r)$ and $f \geqslant 0$. Write $f \in L^{p,\lambda_1}(\mu)$ as $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^C}$, where χ_B is a characteristic function of B. Then we have

$$S := \int_B (K_\alpha f(x))^q \, d\nu(x) \leqslant c\left(\int_B (K_\alpha f_1(x))^q \, d\nu(x) + \int_B (K_\alpha f_2(x))^q \, d\nu(x)\right) := c(S_1 + S_2).$$

Applying Theorem F and the fact $\mu \in (DC)$ we find that

$$S_1 \leqslant \int_X (K_\alpha f_1)^q(x)\, d\nu(x) \leqslant c\left(\int_{B(a,2a_1 r)} (f(x))^p \, d\mu(x)\right)^{q/p}.$$

Now observe that if $\rho(a,x) < r$ and $\rho(a,y) > 2a_1 r$, then $\rho(a,y) > 2a_1\rho(a,x)$. Consequently, using the facts $\mu \in (RDC)$ (see (1)), $0 < \lambda_1 < 1 - \alpha p$ and condition (3) we have

$$\begin{aligned}
S_2 &\leqslant c\int_{B(a,r)} \left(\int_{\rho(a,y)>r} \frac{f(y)}{\mu B(a,\rho(a,y))^{1-\alpha}}\, d\mu(y)\right)^q d\nu(x)\\
&= \nu(B)\left[\sum_{k=0}^{\infty} \int_{B(a,\eta_1^{k+1}r)\setminus B(a,\eta_1^k r)} \frac{f(y)}{\mu B(a,\rho(a,y))^{1-\alpha}}\, d\mu(y)\right]^q\\
&\leqslant c\nu(B)\left[\sum_{k=0}^{\infty}\left(\int_{B(a,\eta_1^{k+1}r)} (f(y))^p\, d\mu(y)\right)^{1/p}\right.\\
&\qquad \left.\times\left(\int_{B(a,\eta_1^{k+1}r)\setminus B(a,\eta_1^k r)} \mu B(a,\rho(a,y))^{(\alpha-1)p'}\, d\mu(y)\right)^{1/p'}\right]^q\\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\nu(B)\left(\sum_{k=0}^{\infty} \mu B(a,\eta_1^{k+1}r)^{\lambda_1/p+\alpha-1+1/p'}\right)^q\\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\nu(B)\mu(B)^{(\lambda_1/p+\alpha-1/p)q}\left(\sum_{k=0}^{\infty}\eta_2^{k(\lambda_1/p+\alpha-1/p)}\right)^q\\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\mu(B)^{q\lambda_1/p} = c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\mu(B)^{\lambda_2},
\end{aligned}$$

where the positive constant c does not depend on B. Now the result follows immediately. □

Proof of Theorem 3.2. *Sufficiency*: Assuming $\alpha = 1/p - 1/q$ and $\mu = \nu$ in Theorem 3.1 we have that K_α is bounded from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\mu)$.

Necessity: Suppose that K_α is bounded from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\mu)$. Then by Theorem 3.1 we have

$$\mu(B)^{1/q-1/p+\alpha} \leqslant c.$$

The conditions $\mu(X)=\infty$ and $\mu\{x\}=0$, for all $x \in X$, implies that $\alpha = 1/p - 1/q$. □

Proof of Theorem 3.3. Let $f \geqslant 0$. For $x, a \in X$, let us introduce the following notation:

$$E_1(x) := \left\{ y : \frac{\rho(a,y)}{\rho(a,x)} < \frac{1}{2a_1} \right\}, \quad E_2(x) := \left\{ y : \frac{1}{2a_1} \leqslant \frac{\rho(a,y)}{\rho(a,x)} \leqslant 2a_1 \right\},$$
$$E_3(x) := \left\{ y : 2a_1 < \frac{\rho(a,y)}{\rho(a,x)} \right\}.$$

For $i = 1, 2, 3$, $r > 0$ and $a \in X$, we denote

$$S_i := \int_{\rho(a,x)<r} \rho(a,x)^\gamma \left(\int_{E_i(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^q d\mu(x).$$

If $y \in E_1(x)$, then $\rho(a,x) < 2a_1 a_0 \rho(x,y)$. Hence, it is easy to see that

$$S_1 \leqslant C \int_B \rho(a,x)^{\gamma+q(\alpha-1)} \left(\int_{\rho(a,y)<\rho(a,x)} f(y)\, d\mu(y) \right)^q d\mu(x).$$

Taking into account the condition $\gamma < (1-\alpha)q - 1$ we have

$$\begin{aligned} \int_{\rho(a,x)>t} \rho(a,x)^{\gamma+q(\alpha-1)}\, d\mu(x) &= \sum_{n=0}^{\infty} \int_{B(a,2^{k+1}t)\setminus B(a,2^k t)} (\rho(a,x))^{\gamma+(\alpha-1)q}\, d\mu(x) \\ &\leqslant c \sum_{n=0}^{\infty} (2^k t)^{\gamma+q(\alpha-1)+1} = ct^{\gamma+q(\alpha-1)+1}, \end{aligned}$$

while the condition $\beta < p-1$ implies

$$\int_{\rho(a,x)<t} \rho(a,x)^{\beta(1-p')+1}\, d\mu(x) \leqslant ct^{\beta(1-p')+1}.$$

Hence

$$\sup_{a\in X, t>0} \left(\int_{\rho(a,x)>t} \rho(a,x)^{\gamma+q(\alpha-1)}\, d\mu(x) \right)^{1/q} \left(\int_{B(a,t)} \rho(a,y)^{\beta(1-p')}\, d\mu(y) \right)^{1/p'} < \infty.$$

Now using Theorem C we have

$$S_1 \leqslant c \left(\int_B \rho(a,x)^\beta (f(y))\, d\mu(y) \right)^{q/p} \leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.$$

Further, observe that if $\rho(a,y) > 2a_1\rho(a,x)$, then $\rho(a,y) \leqslant a_1\rho(a,x) + a_1\rho(a,y) \leqslant \rho(a,y)/2 + a_1\rho(x,y)$. Hence $\rho(a,y)/(2a_1) \leqslant \rho(x,y)$. Consequently, using the growth condition for μ, the fact $\lambda_1 < \beta - \alpha p + 1$ and Lemma E we find that

$$\begin{aligned}
S_3 &\leqslant c\int_{B(a,r)} \rho(a,x)^\gamma \left(\int_{\rho(a,y)>\rho(a,x)} \frac{f(y)}{\rho(a,y)^{1-\alpha}}\,d\mu(y)\right)^q d\mu(x)\\
&\leqslant c\int_{B(a,r)} \rho(a,x)^\gamma \left(\sum_{k=0}^{\infty}\int_{B(a,2^{k+1}\rho(a,x))\backslash B(a,2^k\rho(a,x))} \frac{f(y)}{\rho(a,y)^{1-\alpha}}\,d\mu(y)\right)^q d\mu(x)\\
&\leqslant c\int_{B(a,r)} \rho(a,x)^\gamma \left[\sum_{k=0}^{\infty}\left(\int_{B(a,2^{k+1}\rho(a,x))} f^p(y)\rho(a,y)^\beta\,d\mu(y)\right)^{1/p}\right.\\
&\quad\times\left.\left(\int_{B(a,2^{k+1}\rho(a,x))\backslash B(a,2^k\rho(a,x))} \rho(a,y)^{\beta(1-p')+(\alpha-1)p'}\,d\mu(y)\right)^{1/p'}\right]^q d\mu(x)\\
&\leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)}\int_{B(a,r)} \rho(a,x)^\gamma\\
&\quad\times\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1-\beta/p}(\mu B(a,2^{k+1}\rho(a,x)))^{1/p'}\right)^q d\mu(x)\\
&\leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)}\int_{B(a,r)} \rho(a,x)^\gamma\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1/p-\beta/p}\right)^q d\mu(x)\\
&\leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)}\int_{B(a,r)} \rho(a,x)^{(\lambda_1/p+\alpha-1/p-\beta/p)q+\gamma}\,d\mu(x)\\
&= c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)}\int_{B(a,r)} \rho(a,x)^{\lambda_1 q/p-1}\,d\mu(x) \leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_1 q/p}\\
&= c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.
\end{aligned}$$

So, we conclude that

$$S_3 \leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.$$

To estimate S_2 we consider two cases. First assume that $\alpha < 1/p$. Let

$$E_{k,r} := \{x : 2^k r \leqslant \rho(a,x) < 2^{k+1}r\},$$

$$F_{k,r} := \{x : 2^{k-1}r/a_1 \leqslant \rho(a,x) < a_1 2^{k+2}r\}.$$

Assume that $p^* = p/(1-\alpha p)$. By Hölder's inequality, Corollary B and the assumption $\gamma = q(1/p+\beta/p-\alpha)-1$ we have

$$
\begin{aligned}
S_2 &= \sum_{k=-\infty}^{-1} \int_{E_{k,r}} \rho(a,x)^{\gamma} \left(\int_{E_2(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^{q} d\mu(x) \\
&\leqslant \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma} \left(\int_{E_2(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^{p^*} d\mu(x) \right)^{q/p'} \\
&\quad \times \left(\int_{E_{k,r}} \rho(a,x)^{\gamma p^*/(p^*-q)}\, d\mu(x) \right)^{(p^*-q)/p^*} \\
&\leqslant c \sum_{k=-\infty}^{-1} 2^{k(\gamma+(p^*-q)/p^*)} \left(\int_X I_\alpha(f\chi_{F_{k,r}})(x)^{p^*}\, d\mu(x) \right)^{q/p^*} \\
&\leqslant c \sum_{k=-\infty}^{-1} 2^{k(\gamma+(p^*-q)/p^*)} \left(\int_{F_{k,r}} (f(x))^{p}\, d\mu(x) \right)^{q/p} \\
&\leqslant c \left(\int_{B(a,2a_1 r)} \rho(a,x)^{\beta} (f(x))^{p}\, d\mu(x) \right)^{q/p} \\
&\leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.
\end{aligned}
$$

Let us now consider the case $1/p < \alpha < 1$.

First notice that (see [13])

$$
\int_{E_2(x)} (\rho(x,y)^{(\alpha-1)p'}\, d\mu(y) \leqslant c\rho(a,x)^{1+(\alpha-1)p'},
$$

where the positive constant c does not depend on a and x.

This estimate and Hölder's inequality yield

$$
\begin{aligned}
S_2 &\leqslant c \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma+[(\alpha-1)p'+1)]q/p'} \left(\int_{E_2(x)} (f(y))^{p}\, d\mu(y) \right)^{q/p} d\mu(x) \right)^{q/p'} \\
&\leqslant c \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma+[(\alpha-1)p'+1)]q/p'}\, d\mu(x) \right) \left(\int_{F_{k,r}} (f(y))^{p}\, d\mu(y) \right)^{q/p} \\
&\leqslant c \sum_{k=-\infty}^{-1} (2^k r)^{\gamma+[(\alpha-1)p'+1)]q/p'+1} \left(\int_{F_{k,r}} (f(y))^{p}\, d\mu(y) \right)^{q/p} \\
&= c \sum_{k=-\infty}^{-1} 2^{k\beta q/p} \left(\int_{F_{k,r}} (f(y))^{p}\, d\mu(y) \right)^{q/p} \leqslant c \left(\int_{B(a,2a_1 r)} (f(y))^{p} \rho(a,y)^{\beta}\, d\mu(y) \right)^{q/p} \\
&\leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.
\end{aligned}
$$

Now the result follows immediately. $\square$

Proof of Theorem 3.4. Let $f \geqslant 0$. Suppose that $a \in X$ and $r > 0$. Suppose also that $f_1 = f\chi_{B(a,2a_1r)}$ and $f_2 = f - f_1$. Then $I_\alpha f = I_\alpha f_1 + I_\alpha f_2$. Consequently,

$$\int_{B(a,r)} (I_\alpha f(x))^q \, d\mu(x) \leqslant 2^{q-1} \left(\int_{B(a,r)} (I_\alpha f_1(x))^q \, d\mu(x) + \int_{B(a,r)} (I_\alpha f_2(x))^q \, d\mu(x) \right)$$
$$:= 2^{q-1}(S^{(1)}_{a,r} + S^{(2)}_{a,r}).$$

Due to Theorem A and the condition $s\lambda_1/p = \lambda_2/q$ we have

$$S^{(1)}_{a,r} \leqslant c \left(\int_{B(a,2a_1r)} (f(x))^p \, d\mu(x) \right)^{q/p}$$
$$= c \left(\frac{1}{(2a_1r)^{\lambda_1 s}} \int_{B(a,2a_1r)} (f(x))^p \, dx \right)^{q/p} r^{\lambda_1 sq/p} \leqslant c\|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_2}.$$

Now observe that if $x \in B(a,r)$ and $y \in X \backslash B(a, 2a_1r)$, then $\rho(a,y)/2a_1 \leqslant \rho(x,y)$. Hence Hölder's inequality, condition (2) and the condition $0 < \lambda_1 < p/q$ yield

$$I_\alpha f_2(x) = \int_{X\backslash B(a,2a_1r)} f(y)/\rho(x,y)^{1-\alpha} \, d\mu(y)$$
$$= \sum_{k=0}^{\infty} \left(\int_{B(a,2^{k+2}a_1r)\backslash B(a,2^{k+1}a_1r)} (f(y))^p \, d\mu(y) \right)^{1/p}$$
$$\times \left(\int_{B(a,2^{k+2}a_1r)\backslash B(a,2^{k+1}a_1r)} \rho(a,y)^{(\alpha-1)p'} \, d\mu(y) \right)^{1/p'}$$
$$\leqslant c \sum_{k=0}^{\infty} \left(\frac{1}{(2^{k+1}a_1r)^{\lambda_1 s}} \int_{B(a,2^{k+1}a_1r)} (f(y))^p \, d\mu(y) \right)^{1/p}$$
$$\times (2^k a_1 r)^{\lambda_1 s/p+\alpha-1+s/p'}$$
$$\leqslant c\|f\|_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_1 s/p+\alpha-1+s/p'}.$$

Consequently, by the assumptions $s\lambda_1/p = \lambda_2/q$ and $s = pq(1-\alpha)/(pq+p-q)$ we conclude that

$$S^{(2)}_{a,r} \leqslant c\|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{(\lambda_1 s/p+\alpha-1+s/p')q+s} = c\|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_2}.$$

Summarizing the estimates derived above we finally have the desired result. □

Acknowledgements

The second and third authors were partially supported by the INTAS Grant no. 05-1000008-8157 and the Georgian National Science Foundation Grant no. GNSF/ST07/3-169.

The authors express their gratitude to the referee and Professor Christian Berg for their remarks and suggestions.

References

[1] D.R. Adams, A trace inequality for generalized potentials, Studia Math. 48 (1973) 99–105.

[2] D.R. Adams, A note on Riesz potentials, Duke Math. J. 42 (4) (1975) 765–778.

[3] V. Burenkov, H.V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces, Studia Math. 163 (2) (2004) 157–176.

[4] I.V. Burenkov, H.V. Guliyev, V.S. Guliyev, Necessary and sufficient conditions for the boundedness of the fractional maximal operator in local Morrey-type spaces, Dokl. Akad. Nauk 409 (4) (2006) 443–447 (in Russian).

[5] D. Edmunds, V. Kokilashvili, A. Meskhi, Bounded and Compact Integral Operators, Mathematics and its Applications, vol. 543, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[6] Eridani, V. Kokilashvili, A. Meskhi, Morrey spaces and fractional integral operators, Preprint No. 65, School of Mathematical Sciences, GC University, Lahore, 2007.

[7] J. García-Cuerva, A.E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math. 162 (3) (2004) 245–261.

[8] J. García-Cuerva, J.M. Martell, Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces, Indiana Univ. Math. J. 50 (3) (2001) 1241–1280.

[9] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, Solution of two-weight problems for integral transforms with positive kernels, Georgian Math. J. 3 (1) (1996) 319–342.

[10] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, M. Krbec, Weight Theory for Integral Transforms on Spaces of Homogeneous Type, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 92, Longman, Harlow, 1998.

[11] V. Kokilashvili, Weighted estimates for classical integral operators, in: Nonlinear Analysis, Function Spaces and Applications IV, Roudnice nad Labem, vol. 119, Teubner-Texte Math, Leipzig, 1990, pp. 86–103.

[12] V. Kokilashvili, A. Meskhi, Fractional integrals on measure spaces, Frac. Calc. Appl. Anal. 4 (4) (2001) 1–24.

[13] V. Kokilashvili, A. Meskhi, On some weighted inequalities for fractional integrals on non-homogeneous spaces, Z. Anal. Anwend. 24 (4) (2005) 871–885.

[14] Y. Komori, T. Mizuhara, Notes on commutators and Morrey spaces, Hokkaido Math. J. 32 (2003) 345–353.

[15] J. Peetre, On the theory of $\mathscr{L}^{p,\lambda}$ spaces, J. Funct. Anal. 4 (1969) 71–87.

[16] S. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, London, 1993.

[17] Y. Sawano, H. Tanaka, Morrey spaces for nondoubling measures, Acta Math. Sinica 21 (6) (2005) 1535–1544.

[18] E.M. Stein, G. Weiss, Fractional integrals on n-dimensional Euclidean spaces, J. Math. Mech. 7 (4) (1958) 503–514.

[19] J.O. Strömberg, A. Torchinsky, Weighted Hardy Spaces, Lecture Notes in Mathematics, vol. 1381, Springer, Berlin, 1989.

Expositiones Mathematicae

Country Netherlands - SIR Ranking of Netherlands

Subject Area and Category Mathematics
Mathematics (miscellaneous)

Publisher Elsevier BV

Publication type Journals

ISSN 07230869

Coverage 2004-ongoing

Scope Our aim is to publish papers of interest to a wide mathematical audience, such as expository articles that make research results more widely accessible, and papers on topics with fairly broad appeal. Main articles must be written in such a way that a research student interested in the topic of the paper can read them profitably. Mathematical notes can be at a slightly higher level. Clarity of exposition, accuracy of details and the relevance and interest of the subject matter will be the decisive factors in our acceptance of an article for publication. All paper are refereed.

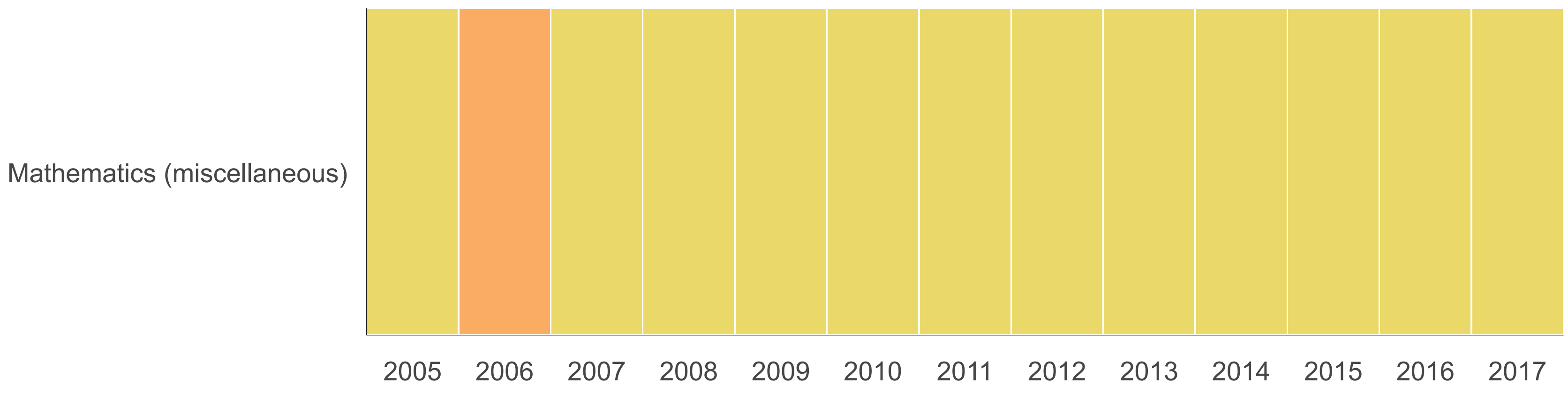
? Homepage

19

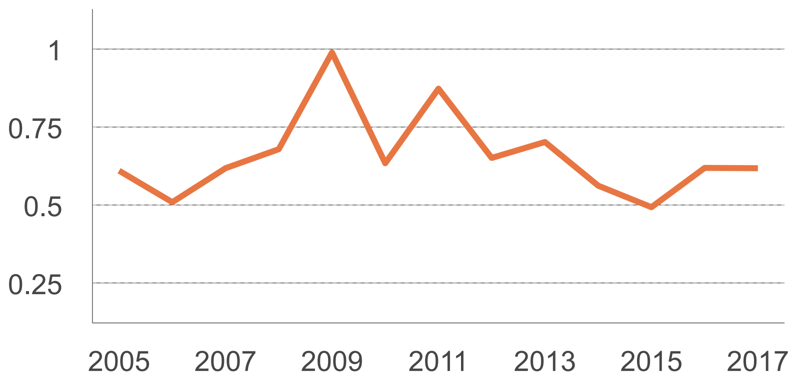
H Index

Join the conversation about this journal

Quartiles

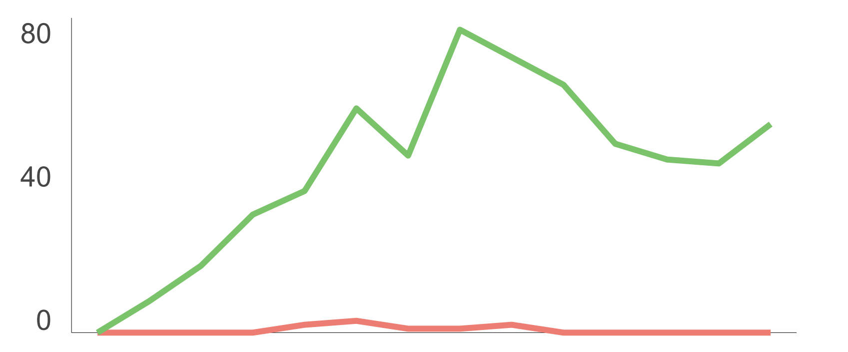


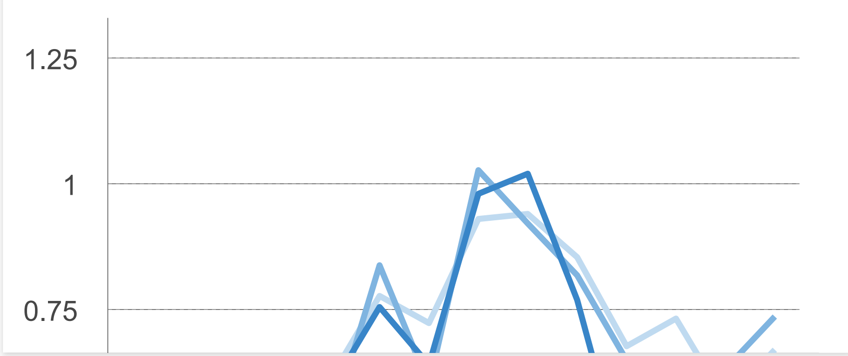
SJR

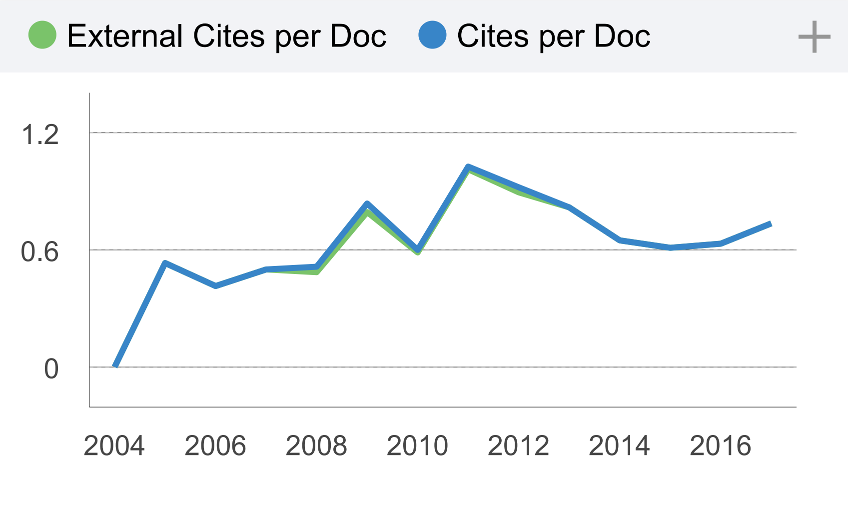


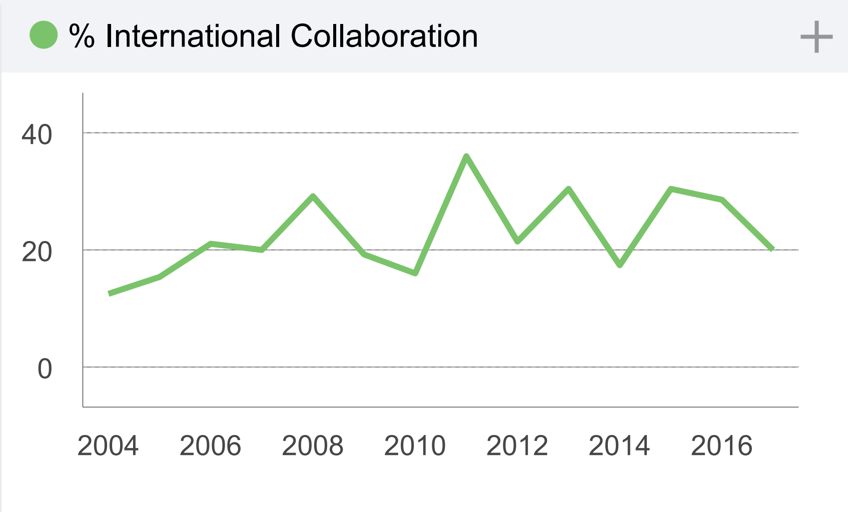
Citations per document

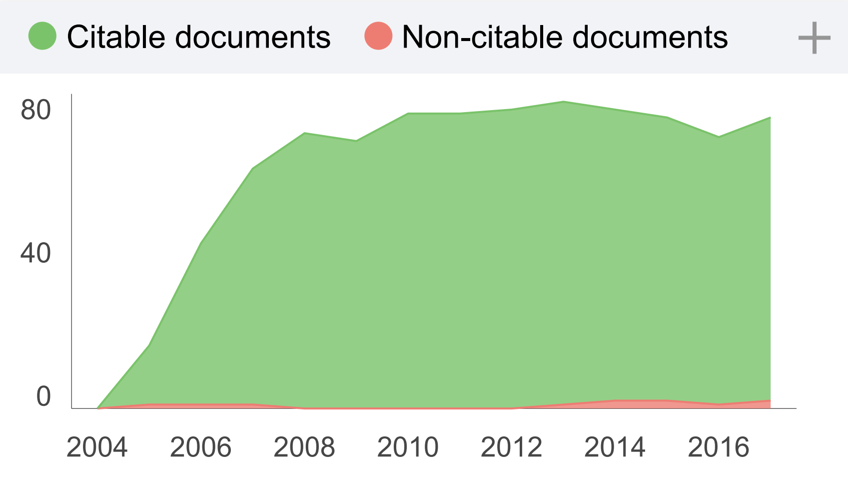
Total Cites Self-Cites

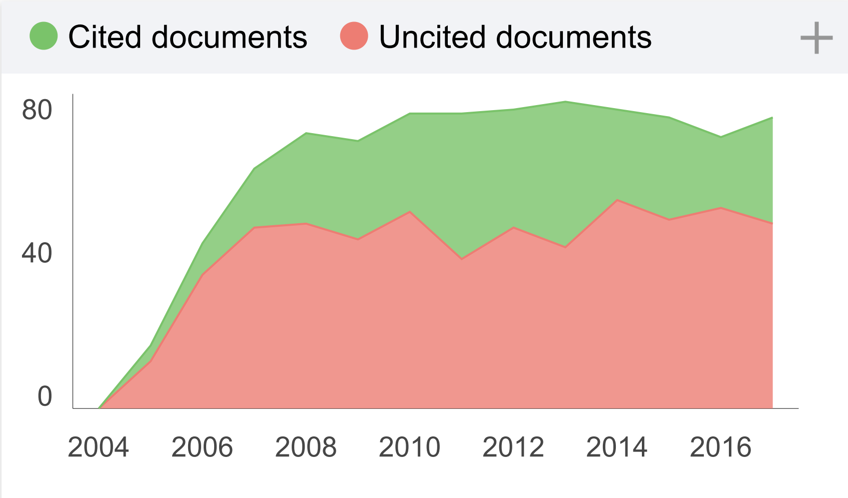

80
40
0


1.25
1
0.75


External Cites per Doc
Cites per Doc
1.2
0.6
0
2004
2006
2008
2010
2012
2014
2016


% International Collaboration
40
20
0
2004
2006
2008
2010
2012
2014
2016


Citable documents
Non-citable documents
80
40
0
2004
2006
2008
2010
2012
2014
2016


Cited documents
Uncited documents
80
40
0
2004
2006
2008
2010
2012
2014
2016

← Show this widget in your own website

Just copy the code below and paste within your html code:

<a href="https://www.scimag

Leave a comment

Name

Email

(will not be published)

Submit

The users of Scimago Journal & Country Rank have the possibility to dialogue through comments linked to a specific journal. The purpose is to have a forum in which general doubts about the processes of publication in the journal, experiences and other issues derived from the publication of papers are resolved. For topics on particular articles, maintain the dialogue through the usual channels with your editor.

Developed by: Powered by:

Follow us on @ScimagoJR

EST MODUS IN REBUS

Horatio (Satire 1,1,106)