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Submission date: 15-Apr-2019 05:08PM (UTC+0800)

Submission ID: 1112745475

File name: 5_Q3_0.25_3_IJCIET_10_01_209.pdf (813.19K)

Word count: 1614

Character count: 10002

THE FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES OVER Q -HOMOGENEOUS METRIC MEASURE SPACE

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ABSTRACT

This paper establishes necessary and sufficient condition for the boundedness of the fractional integral operator $I_{\alpha f}$ on Morrey spaces over metric measure spaces which satisfies the Q -homogeneous and its corollary.

Key words: Morrey Space Classic; Metric Measure Space; Q -Homogeneous.

Cite this Article: Hairur Rahman, M. Imam Utoyo and Eridani, The Fractional Integral Operators on Morrey Spaces Over Q -Homogeneous Metric Measure Space, *International Journal of Civil Engineering and Technology (IJCET)* 10(1), 2019, pp. 2309–2322.

<http://www.iaeme.com/IJCET/issues.asp?JType=IJCET&VType=10&IType=1>

1. INTRODUCTION

We consider to a topological space $X := (X, \delta, \mu)$, endowed with complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (metric) $\delta: X \times X \rightarrow [0, \infty)$ satisfying the following conditions.

1. $\delta(x, y) = 0$ if and only if $x = y$;
2. $\delta(x, y) > 0$ for all $x \neq y, x, y \in X$;
3. $\delta(x, y) = \delta(y, x)$;
4. $\delta(x, y) \leq \{\delta(x, z) + \delta(z, y)\}$

for every $x, y, z \in X$. We have an assumptions that the balls $B(a, r) := \{x \in X: \delta(x, a) < r\}$ are measurable, for $a \in X, r > 0$, and $0 \leq \mu(B(a, r)) < \infty$. For every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty$, $\mu\{a\} =$

0 and $B(a, r_2) \setminus B(a, r_1) = \emptyset$, for all $a \in X, 0 < r_1 < r_2 < \infty$. The triple (X, δ, μ) will be called metric measure space [7].

X is called Q -homogeneous ($Q > 0$) such that $C_0 r^Q \leq \mu(B) \leq C_1 r^Q$ where C_0 and C_1 are positive constants [8].

Eridani [6,7] proved the boundedness theorem on Lebesgue spaces in K_α and classic Morrey spaces over quasi metric space where

$$K_\alpha := \int_X \frac{f(y)}{\mu(B(x, \delta(x, y)))^{1-\alpha}} d\mu(y)$$

with $0 < \alpha < 1$.

The result of [7] can be adapted to the operator K_α with doubling condition. Let $0 < \alpha < \beta$, we consider the fractional integral operator I_α given by

$$I_\alpha f(x) := \int_X \frac{f(y)}{\delta(x, y)^{\beta-\alpha}} d\mu(y)$$

for suitable f on X

The boundedness theorem of I_α on homogeneous classic Morrey spaces can be proved using Q -Homogeneous. In this paper, we will prove the generalization of the boundedness theorem from [6,7].

2. PRELIMINARIES

The following theorem is the inequality for the operator K_α from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ for the case of Euclidean spaces.

Theorem 2.1 [6] Let (X, δ, μ) be a space of homogeneous type. Suppose that $1 < p < q < \infty$ and $0 < \alpha < \frac{1}{p}$. Assume that ν is another measure on X . Then K_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ if and only if

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\alpha)}$$

for all balls B in X .

Eridani and Meshki [7] proved the boundedness results of K_α from $\mathcal{L}^p(X, \mu)$ to the classic Morrey spaces $\mathcal{L}^{p,\lambda}(X, \nu, \mu)$ which is defined as a set of functions $f \in \mathcal{L}_{loc}^p(X, \nu)$ such that

$$\|f: \mathcal{L}^{p,\lambda}(X, \nu, \mu)\| = \sup_B \left(\frac{1}{\mu(B)^\lambda} \int_B |f(y)|^p d\nu(y) \right)^{\frac{1}{p}} < \infty.$$

with ν is another measure on X , where $1 \leq p < \infty$ and $\lambda \geq 0$. Their theorem can be stated as the following theorem.

Theorem 2.2 [7] Let (X, δ, μ) be a space of homogeneous type and let $1 < p < q < \infty$. Suppose that $0 < \alpha < \frac{1}{p}$, $0 < \lambda_1 < 1 - \alpha p$ and $\frac{\lambda_2}{q} = \frac{\lambda_1}{p}$. Then K_α is bounded from $\mathcal{L}^{p,\lambda_1}(X, \nu, \mu)$ to $\mathcal{L}^{q,\lambda_2}(X, \nu, \mu)$ if and only if there is a positive constant C such that

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\alpha)}$$

3. MAIN RESULT

In this section, we formulate the main results of the paper. We begin with the case of β -homogeneous over metric measure space.

Theorem 3.1 Let (X, δ, μ) be a β -homogeneous metric measure space, ν be a measure on X , $1 < p < q < \infty$, $1 < \alpha < \beta$. Then I_α is bounded from $L^p(X, \mu)$ to $L^q(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\frac{\alpha}{\beta})}$$

Proof:(Necessity) If $x, y \in B(a, r)$ then $\delta(x, a) < r$ and $\delta(y, a) < r$ thus $\delta(x, y) \leq \delta(x, a) + \delta(y, a) < 2r$ thus

$$\frac{1}{(2r)^{\beta-\alpha}} \leq \frac{1}{\delta(x, y)^{\beta-\alpha}}$$

the above inequality implies.

$$\begin{aligned} \frac{\mu(B)}{r^{\beta-\alpha}} &= \int_B \frac{d\mu(y)}{(2r)^{\beta-\alpha}} \leq \int_B \frac{d\mu(y)}{\delta(x, y)^{\beta-\alpha}} = \int_X \frac{\chi_B(y)d\mu(y)}{\delta(x, y)^{\beta-\alpha}} = CI_\alpha \chi_B(x) \\ r^\alpha &\leq CI_\alpha \chi_B(x) \end{aligned}$$

$$\begin{aligned} \|I_\alpha \chi_B : L^q(\nu)\| &\leq C \|\chi_B : L^p(\mu)\| \leq C \left(\int_X \chi_B(t)d\mu(t) \right)^{\frac{1}{p}} \leq C\mu(B)^{\frac{1}{p}} \\ \left(\int_B |r^\alpha|^q d\nu(x) \right)^{\frac{1}{q}} &\leq C \left(\int_B |I_\alpha \chi_B(t)|^q d\nu(t) \right)^{\frac{1}{q}} \leq C \|I_\alpha \chi_B : L^q(\nu)\| \leq C\mu(B)^{\frac{1}{p}} \end{aligned}$$

Thus

$$r^\alpha \nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p}}$$

$C_0 r^\beta \leq \mu(B) \leq C_1 r^\beta$ thus

$$\begin{aligned} \mu(B)^{\frac{\alpha}{\beta}} &\leq Cr^\alpha \\ \mu(B)^{\frac{\alpha}{\beta}} \nu(B)^{\frac{1}{q}} &\leq Cr^\alpha \nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p}} \end{aligned}$$

$$\nu(B)^{\frac{1}{q}} \mu(B)^{\frac{\alpha-1}{\beta-p}} \leq C$$

Thus

$$\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1-\alpha}{p-\beta}}$$

or alternatively

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\frac{\alpha}{\beta})}$$

Sufficiency: Let $f \geq 0$. We define

$$S(s) := \int_{\delta(a,y) < s} f(y)d\mu(y)$$

for every $s \in [0, r]$. Suppose that $S(r) < \infty$, then $2^m < S(r) \leq 2^{m+1}$, for some $m \in \mathbb{Z}$.

Let

$$s_j := \sup\{t : S(t) \leq 2^j\}, j \leq m, \text{ and } s_{m+1} := r.$$

Then $(s_j)_{j=-\infty}^{m+1}$ is non-decreasing sequence, $S(s_j) \leq 2^j, S(t) \geq 2^j$ for $t > s_j$ and

$$2^j \leq \int_{s_j \leq \delta(a,y) \leq s_{j+1}} f(y) d\mu(y)$$

If $\rho := \lim_{j \rightarrow -\infty} s_j$, then

$$\delta(a,x) < r \Leftrightarrow \delta(a,x) \in [0, \rho] \cup \bigcup_{j=-\infty}^m (s_j, s_{j+1}],$$

if $S(r) = \infty$ then $m = \infty$. Thus

$$0 \leq \int_{\delta(a,y) < \rho} f(y) d\mu(y) \leq S(s_j) \leq 2^j$$

for every j , thus

$$\int_{\delta(a,y) < \rho} f(y) d\mu(y) = 0$$

from these observations, we have

$$\begin{aligned} \int_{\delta(a,x) < r} (I_\alpha f(x))^q d\nu(x) &\leq \sum_{j=-\infty}^m \int_{s_j \leq \delta(a,x) \leq s_{j+1}} (I_\alpha f(x))^q d\nu(x) \\ &\leq \sum_{j=-\infty}^m \int_{s_j \leq \delta(a,x) \leq s_{j+1}} \left(\int_{\delta(a,y) \leq s_{j+1}} \frac{f(y) d\mu(y)}{\delta(x,y)^{\beta-\alpha}} \right)^q d\nu(x) \\ &\leq \sum_{j=-\infty}^m \int_{s_j \leq \delta(a,x) \leq s_{j+1}} \left(\sum_{k=0}^{\infty} \left(\frac{1}{s_{j+1}} \right)^{\beta-\alpha} \int_{\delta(a,y) \leq s_{j+1}} f(y) d\mu(y) \right)^q d\nu(x) \\ &\leq \left(\sum_{j=-\infty}^m \left(\frac{1}{s_{j+1}} \right)^{\beta-\alpha} \int_{\delta(a,y) \leq s_{j+1}} f(y) d\mu(y) \right)^q \nu(B) \end{aligned}$$

Using the fact that

$$\int_{\delta(a,y) \leq s_{j+1}} f(y) d\mu(y) \leq S(s_{j+1}) \leq 2^{j+2} \leq C \int_{s_{j-1} \leq \delta(a,y) \leq s_j} f(y) d\mu(y)$$

then, by using Holder's inequality, we obtain

$$\begin{aligned} &\leq C \nu(B) \left(\sum_{j=-\infty}^m \left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} (f(y))^p d(\mu)y \right)^{\frac{1}{p}} \left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} 1^q d(\mu)y \right)^{\frac{1}{q}} \frac{1}{s_j^{\beta-\alpha}} \right)^q \\ &\leq C \nu(B) \left(\left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} (f(y))^p d(\mu)y \right)^{\frac{1}{p}} \sum_{j=-\infty}^m \mu(B(x,r))^{1-\frac{1}{p}} \frac{1}{s_j^{\beta-\alpha}} \right)^q \\ &= C \nu(B) r^{q(\alpha - \frac{\beta}{p})} \left(\left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} (f(y))^p d(\mu)y \right)^{\frac{1}{p}} \right)^q \end{aligned}$$

$$\begin{aligned} &\leq C\mu(B)^{q(\frac{1}{p}-\frac{\alpha}{\beta})}r^{q(\alpha-\frac{\beta}{p})}\left(\left(\int_{s_{j-1}\leq \delta(a,y)\leq s_j}(f(y))^pd(\mu)y\right)^{\frac{1}{p}}\right)^q \\ &= C\left(\left(\int_{s_{j-1}\leq \delta(a,y)\leq s_j}(f(y))^pd(\mu)y\right)^{\frac{1}{p}}\right)^q \end{aligned}$$

Thus

$$\|I_\alpha f : \mathcal{L}^q(\nu)\| \leq C\|f : \mathcal{L}^p(\mu)\|$$

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Next, using the modified condition for measure ν , we obtain the following result.

Theorem 3.2 Let (X, δ, μ) be a Q-homogeneous metric measure space, ν be a measure on X , $1 < p < q < \infty$, $1 < \alpha < \beta - \frac{Q}{p'}$. Then I_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\nu(B) \leq Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$$

with $p' = \frac{p}{p-1}$.

Proof. (Necessity) Suppose that I_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ thus

$$\|I_\alpha f : \mathcal{L}^q(X, \nu)\| \leq C\|f : \mathcal{L}^p(X, \mu)\|$$

Hence,

$$\left(\int_X |I_\alpha f|^q d\nu\right)^{1/q} \leq C \left(\int_X |f(x)|^p d\mu\right)^{1/p}$$

$f := \chi_B$ where $a \in X, r > 0$ then

$$\left(\int_B |I_\alpha \chi_B|^q d\nu\right)^{1/q} \leq C \left(\int_B |\chi_B|^p d\mu\right)^{1/p}$$

$$\left(\int_B \left(\int_B \frac{\chi_B}{\delta(x, y)^{\beta-\alpha}} d\mu(y)\right)^q d\nu\right)^{1/q} \leq C\mu(B)^{1/p}$$

$$\begin{aligned} r^{\alpha-\beta}\mu(B)\nu(B)^{1/q} &\leq C\mu(B)^{1/p} \\ \nu(B)^{1/q} &\leq C\mu(B)^{\frac{1}{p}-1}r^{\beta-\alpha} \end{aligned}$$

Because $p' = \frac{p}{p-1}$ and $C_0 r^Q \leq \mu(B) \leq C_1 r^Q$ then

$$\nu(B)^{1/q} \leq Cr^{-\frac{Q}{p'}} r^{\beta-\alpha}$$

$$\nu(B) \leq Cr^{q(\beta-\alpha-\frac{q}{p'})}$$

Sufficiency. Let $f \geq 0$. For $x, a \in X$, next we consider the notation

$$E_1(x) := \left\{ y : \delta(a, y) < \frac{\delta(a, x)}{2a_1} \right\};$$

$$E_2(x) := \left\{ y : \frac{\delta(a, x)}{2a_1} \leq \delta(a, y) \leq 2a_1\delta(a, x) \right\};$$

$$E_3(x) := \{y : \delta(a, y) > a_1\delta(a, x)\}.$$

Thus

$$\begin{aligned} \int_X (I_\alpha f(x)) d\nu(x) &\leq C \int_X \left(\int_{E_1(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &\quad + C \int_X \left(\int_{E_2(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &\quad + C \int_X \left(\int_{E_3(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) = S_1 + S_2 + S_3 \end{aligned}$$

If $y \in E_1(x)$, then $\delta(a, x) < 2a_1a_0\delta(a, x)$. Thus obviously

$$\begin{aligned} S_1 &= \int_{\delta(a,x) < r} \left(\int_{E_1(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &\leq C \int_B \left(\int_{\delta(a,y) < \delta(a,x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &\leq C \int_B \delta(a, x)^{q(\alpha-\beta)} \left(\int_{\delta(a,y) < \delta(a,x)} |f(y)| d\mu(y) \right)^q d\nu(x) \end{aligned}$$

Thus we have

$$\begin{aligned} \int_{\delta(a,x) \geq t} \delta(a, x)^{q(\alpha-\beta)} d\nu(x) &= \sum_{n=0}^{\infty} \int_{B(a, 2^{k+1}t) \setminus B(a, 2^k t)} (\delta(a, x)^{q(\alpha-\beta)} d\nu(x)) \\ &\leq C \sum_{n=0}^{\infty} (2^k t)^{q(\alpha-\beta)} \nu(B) = Ct^{q(\alpha-\beta)} \nu(B) \end{aligned}$$

which implies

$$\int_{\delta(a,x) \leq t} 1^{(1-p')} d\mu(x) \leq C\mu(B)$$

Thus

$$\begin{aligned}
 & \sup_{a \in X, t > 0} \left(\int_{\delta(a,x) \geq t} \delta(a,x)^{q(\alpha-\beta)} dv(x) \right)^{\frac{1}{q}} \left(\int_{\delta(a,x) \leq t} 1^{(1-p)} d\mu(x) \right)^{\frac{1}{p}} \\
 & \leq \left(C t^{q(\alpha-\beta)} v(B) \right)^{\frac{1}{q}} C \mu(B)^{\frac{1}{p}} \\
 & \leq C t^{(\alpha-\beta)} C t^{\left(\beta - \alpha - \frac{Q}{p} \right) q \frac{1}{q}} t^{Q \left(\frac{p-1}{p} \right)} = C < \infty
 \end{aligned}$$

Now, using theorem C in [9], we have

$$S_1 \leq C \left(\int_B |f(y)|^p d\mu(y) \right)^{q/p} \leq C \|f\|_{L^p(X,\mu)}^q$$

Next, we observe that if $\delta(a,y) > 2a_1\delta(a,x)$, then $\delta(a,y) \leq a_1\delta(a,x) + a_1\delta(a,y) \leq \delta(a,y)/2 + a_1\delta(x,y)$. Thus $\delta(a,y)/2a_1 \leq \delta(x,y)$. Implies, using the condition $v(B) \leq Cr^{\left(\beta - \alpha - \frac{Q}{p} \right) q}$, then

$$\begin{aligned}
 S_3 & \leq C \int_{B(a,r)} \left(\int_{\delta(a,y) > \delta(a,x)} \frac{|f(y)|}{\delta(a,y)^{\beta-\alpha}} d\mu(y) \right)^q dv(x) \\
 & \leq C \int_{B(a,r)} \left(\sum_{k=0}^{\infty} \int_{B(a,2^{k+1}\delta(a,x)) \setminus B(a,2^k\delta(a,x))} \frac{|f(y)|}{\delta(a,y)^{\beta-\alpha}} d\mu(y) \right)^q dv(x) \\
 & \leq C \int_{B(a,r)} \left[\sum_{k=0}^{\infty} \left(\int_{B(a,2^{k+1}\delta(a,x))} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left. \left(\int_{B(a,2^{k+1}\delta(a,x)) \setminus B(a,2^k\delta(a,x))} \delta(a,y)^{(\alpha-\beta)p} d\mu(y) \right)^{\frac{1}{p}} \right]^q dv(x) \\
 & \leq C \|f\|_{L^p(X,\mu)}^q \int_{B(a,r)} \left(\sum_{k=0}^{\infty} (2^k \delta(a,x))^{\alpha-\beta} (\mu B(a,2^{k+1}\delta(a,x)))^{\frac{1}{p}} \right)^q dv(x) \\
 & \leq C \|f\|_{L^p(X,\mu)}^q \int_{B(a,r)} \left(\sum_{k=0}^{\infty} (2^k \delta(a,x))^{\alpha-\beta} r^{\frac{Q}{p}} \right)^q dv(x) \\
 & = C \|f\|_{L^p(X,\mu)}^q r^{(\alpha-\beta)q} r^{\frac{Qq}{p}} v(B) \\
 & = C \|f\|_{L^p(X,\mu)}^q
 \end{aligned}$$

Hence, we conclude that

$$S_3 \leq C \|f\|_{L^p(X,\mu)}^q$$

To estimate S_2 , we consider two cases. First assumption is that $\alpha < \beta - \frac{Q}{p}$. The hypothesis on the theorem $\alpha > 0$ which implies $0 < \alpha < \beta - \frac{Q}{p}$. Given $p^* = \frac{pQ}{p(\beta-\alpha-Q)+Q}$ then $q \leq p^*$. First assumption $q < p^*$ and suppose that

$$F_k := \{x : 2^k \leq \delta(a, x) < 2^{k+1}\};$$

$$G_k := \left\{y : \frac{2^{k-2}}{a_1} \leq \delta(a, y) \leq a_1 2^{k+2}\right\}.$$

Assume that $\frac{p^*}{q}$, using Holder's inequality, we obtain ⁶

$$\begin{aligned} S_2 &= \int_X \left(\int_{E_2(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &= C \sum_{k \in \mathbb{Z}} \int_{F_k} \left(\int_{E_2(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q d\nu(x) \\ &\leq \sum_{k \in \mathbb{Z}} \left(\int_{F_k} \left(\int_{E_2(x)} |f(y)| \delta(a, x)^{\alpha-\beta} d\mu(y) \right)^{p^*} d\nu(x) \right)^{\frac{q}{p^*}} \times \left(\int_{F_k} 1^{\frac{p^*}{p^*-q}} d\nu(x) \right)^{\frac{p^*-q}{p^*}} \\ &\leq C \sum_{k \in \mathbb{Z}} \nu(B)^{\frac{p^*-q}{p^*}} \left(\int_X \left(I_\alpha(|f| \chi_{G_k}) \right)^{p^*} d\nu(y) \right)^{\frac{q}{p^*}} \\ &\leq C \sum_{k \in \mathbb{Z}} \nu(B)^{\frac{p^*-q}{p^*}} \left(\int_{G_k} |f(y)|^p d\mu(y) \right)^{\frac{q}{p}} \end{aligned}$$

Where

$$\begin{aligned} \frac{p^* - q}{p^*} &= 1 - \frac{q}{p^*} \\ &= 1 - \frac{q(p(\beta - \alpha) - pQ + Q)}{pQ} \\ &= 1 - \frac{Q(pq + p - q)}{pQ} + q - \frac{q}{p} = 0 \end{aligned}$$

$$\begin{aligned} &\leq C \left(\int_{G_k} |f(y)|^p d\mu(x) \right)^{\frac{q}{p}} \\ &\leq C \|f\|_{L^p(X, \mu)}^q \end{aligned}$$

if $q = p^*$, thus, we have

$$\begin{aligned} S_2 &= \int_X \left(\int_{E_2(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^q dv(x) \\ &= C \sum_{k \in Z} \int_{F_k} \left(\int_{E_2(x)} |f(y)| \delta(x, y)^{\alpha-\beta} d\mu(y) \right)^{p^*} dv(x) \\ &\leq C \sum_{k \in Z} \left(\int_X (I_\alpha(|f| \chi_{G_k})) dv(y) \right)^{p^*} \\ &\leq C \sum_{k \in Z} \left(\int_{G_k} |f(y)|^p d\mu(x) \right)^{\frac{q}{p}} \\ &\leq C \left(\int_{G_k} |f(y)|^p d\mu(x) \right)^{\frac{p^*}{p}} \\ &\leq C \|f\|_{L^p(X, \mu)}^q \end{aligned}$$

If $\alpha > \beta - \frac{q}{p}$, ⁶ using Holder's inequality, we obtain

$$S_2 \leq \int_X \left(\int_{E_2(x)} (f(y))^p d\mu(y) \right)^{\frac{q}{p}} \left(\int_{E_2(x)} \delta(a, x)^{(\alpha-\beta)p'} d\mu(y) \right)^{\frac{q}{p'}} dv(y)$$

thus we have

$$\begin{aligned} \int_{E_2(x)} \delta(a, x)^{(\alpha-\beta)p'} d\mu(y) &\leq \int_0^\infty \mu(B(a, \delta(a, x)) \cap \{y | \delta(x, y) < \lambda^{\frac{1}{(\alpha-\beta)p}}\}) d\lambda \\ &\leq \int_0^{\delta(a, x)^{(\alpha-\beta)p'}} \mu(B(a, \delta(a, x)) \cap \{y | \delta(x, y) < \lambda^{\frac{1}{(\alpha-\beta)p}}\}) d\lambda \\ &\quad + \int_{\delta(a, x)^{(\alpha-\beta)p'}}^\infty \mu(B(a, \delta(a, x)) \cap \{y | \delta(x, y) < \lambda^{\frac{1}{(\alpha-\beta)p}}\}) d\lambda \\ &\leq C \delta(a, x)^{Q+(\alpha-\beta)p'} + \int_{\delta(a, x)^{(\alpha-\beta)p'}}^\infty \lambda^{\frac{1}{(\alpha-\beta)p}} d\lambda = C \delta(a, x)^{Q+(\alpha-\beta)p'} \end{aligned}$$

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where the positive constant C is independent of a and x . Hence, using Holder's inequality, we obtain

$$\begin{aligned} S_2 &\leq \int_X \left(\int_{E_2(x)} \delta(a, x)^{(\alpha-\beta)p'} d\mu(y) \right)^{\frac{q}{p}} \left(\int_{E_2(x)} |f(y)|^p d\mu(y) \right)^{\frac{q}{p}} dv(x) \\ &\leq \sum_{k \in \mathbb{Z}} \int_{F_k} \delta(a, x)^{q+(\alpha-\beta)p'(\frac{q}{p})} \left(\int_{E_2(x)} |f(y)|^p d\mu(y) \right)^{\frac{q}{p}} dv(y) \\ &\leq C 2^{k((\beta-\alpha-\frac{q}{p})q + \frac{q}{p} + (\alpha-\beta)q)} \left(\int_{G_k} |f(y)|^p d\mu(y) \right)^{\frac{q}{p}} \leq C \left(\int_X |f(y)|^p d\mu(y) \right)^{\frac{q}{p}} \\ &\leq C \|f\|_{L^p(X, \mu)}^q \end{aligned}$$

The proof is complete.

The similar results concerning the boundedness properties of the fractional integral operator I_α on the classic Morrey spaces using Q -homogeneous metric measure space is obtained by the following theorem.

Theorem 3.3 Let (X, δ, μ) be a Q -homogeneous metric measure space, ν be a measure on X , $1 < p < q < \infty$, $1 < \alpha < \beta - \frac{q}{p'}$, $0 < \lambda_1 < \frac{\beta p}{q}$, and $\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$. Then I_α is bounded from $L^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)$ to $L^{q, \frac{\lambda_2}{q}}(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\nu(B) \leq Cr^{(\beta-\alpha-\frac{q}{p'})q}$$

with $p' = \frac{p}{p-1}$.

Proof: (Necessity) Suppose that I_α is bounded from $L^{p, \frac{Q\lambda_1}{\beta}}(X, \mu)$ to $L^{q, \lambda_2}(X, \nu)$ which implies that

$$\|I_\alpha f: L^{q, \lambda_2}(X, \nu)\| \leq C \|f: L^{p, \frac{Q\lambda_1}{\beta}}(X, \mu)\|$$

Thus

$$\left(\frac{1}{\mu(B)^{\lambda_2}} \int_X |I_\alpha f|^q d\nu(x) \right)^{\frac{1}{q}} \leq C \left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}} \int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

$f := \chi_B$ where $a \in X$ and $r > 0$ then

$$\left(\frac{1}{\mu(B)^{\lambda_2}} \int_X |I_\alpha \chi_B(x)|^q d\nu(x) \right)^{\frac{1}{q}} \leq C \left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}} \int_X |\chi_B(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

$$\left(\frac{1}{\mu(B)^{\lambda_2}} \int_B \left(\int_B \frac{\chi_B}{\delta(x,y)^{\beta-\alpha}} d\mu(y) \right)^q d\nu(x) \right)^{\frac{1}{q}} \leq C \mu(B)^{\frac{-Q\lambda_1}{p\beta}} \mu(B)^{\frac{1}{p}}$$

$$\mu(B)^{\frac{-\lambda_2}{q}} r^{\alpha-\beta} \mu(B) v(B)^{\frac{1}{q}} \leq C \mu(B)^{\frac{-Q\lambda_1}{p\beta}} \mu(B)^{\frac{1}{p}}$$

Because $p' = \frac{p}{p-1}$, $\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$ and $C_0 r^Q \leq \mu(B) \leq C_1 r^Q$ then

$$v(B)^{\frac{1}{q}} \leq C \mu(B)^{\frac{1}{p'}} r^{\alpha-\beta}$$

$$v(B)^{\frac{1}{q}} \leq C r^{-\frac{Q}{p'}} r^{\beta-\alpha}$$

$$v(B) \leq C r^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$$

Sufficiency. Given arbitrary ball B on X . Suppose that $B := B(a, r)$ and $\tilde{B} := (a, 2r)$ and $f \in L^{p, \frac{Q\lambda_1}{\beta}}(\mu)$. we write

$$f = f_1 + f_2 := f_{\chi_{\tilde{B}}} + f_{\chi_{\tilde{B}}^c}$$

$$\|f_1: L^p(\mu)\| = \left(\int_B |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

$$= \mu(B)^{\frac{Q\lambda_1}{\beta p}} \left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}} \int_B |f(x)|^p d\mu \right)^{\frac{1}{p}}$$

$$\leq \mu(B)^{\frac{Q\lambda_1}{\beta p}} \|f: L^{p, \frac{Q\lambda_1}{\beta}}(X, \mu)\|$$

if $f_1 \in L^p(X, \mu)$, and using Theorem 3.2, it is obvious that

$$\begin{aligned} \left(\frac{1}{\mu(B)^{\lambda_2}} \int_B |I_\alpha f_1|^q d\nu(x) \right)^{\frac{1}{q}} &\leq \mu(B)^{-\frac{\lambda_2}{q}} \left(\int_B |I_\alpha f_1|^q d\nu(x) \right)^{\frac{1}{q}} \\ &\leq \mu(B)^{-\frac{\lambda_2}{q}} \|I_\alpha f_1: L^q(\nu)\| \\ &\leq C \mu(B)^{-\frac{\lambda_2}{q}} \|f_1: L^p(\mu)\| \end{aligned}$$

$$\leq C\mu(B)^{-\frac{\lambda_2}{q}}\mu(B)^{-\frac{Q\lambda_1}{\beta p}}\left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta}}(X, \mu)\right\|$$

$$\leq C\left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta}}(X, \mu)\right\|$$

further we will prove,

$$\begin{aligned}|I_\alpha f_2(x)| &= \left|\int_{\delta(x,y) \geq r} \frac{f(y)}{\delta(x,y)^{\beta-\alpha}} d\mu(y)\right| \\&\leq \int_{\delta(x,y) \geq r} \frac{|f(y)|}{\delta(x,y)^{\beta-\alpha}} d\mu(y) \\&\leq \sum_{k=0}^{\infty} \int_{2^k r \leq \delta(x,y) \leq 2^{k+1}r} \frac{|f(y)|}{\delta(x,y)^{\beta-\alpha}} d\mu(y) \\&\leq \sum_{k=0}^{\infty} \left(\frac{1}{2^k r}\right)^{\beta-\alpha} \int_{\delta(x,y) \leq 2^{k+1}r} |f(y)| d\mu(y) \\&\leq C \sum_{k=0}^{\infty} \left(\int_{\delta(x,y) \leq 2^{k+1}r} |f(x)|^p d\mu(y)\right)^{\frac{1}{p}} \left(\int_{\delta(x,y) \leq 2^{k+1}r} 1^q d\mu(y)\right)^{\frac{1}{q}} \frac{1}{2^k r^{\beta-\alpha}} \\&\leq C\mu(B)^{\frac{Q\lambda_1}{\beta p}} \left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)\right\| \sum_{k=0}^{\infty} \frac{1}{\mu(B(x, 2^{k+1}r))^{\frac{1}{q}}} \frac{1}{(2^k r)^{\beta-\alpha}} \\&= \mu(B)^{\frac{Q\lambda_1}{\beta p}} r^{\alpha-\beta} r^{\frac{Q}{p'}} \left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)\right\|\end{aligned}$$

Then

$$\begin{aligned}\left(\frac{1}{\mu(B)^{\lambda_2}} \int_B |I_\alpha f_2(x)|^q d\nu(x)\right)^{\frac{1}{q}} &= C\nu(B)^{\frac{1}{q}} \mu(B)^{\frac{-\lambda_2}{q}} \mu(B)^{\frac{Q\lambda_1}{\beta p}} r^{\alpha-\beta} r^{\frac{Q}{p'}} \left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)\right\| \\&= C\left\|f: \mathcal{L}^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)\right\|\end{aligned}$$

The proof is complete.

The condition $\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$ is interchangeable to the condition $\nu(B) \leq Cr^{q(\beta-\alpha-\frac{Q}{p'})}$. Yet, the following theorem is hold obviously.

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Theorem 3.4 Let (X, δ, μ) be a Q -homogeneous metric measure space, ν be a measure on X ,

$1 < p < q < \infty$, $1 < \alpha < \beta - \frac{q}{p}$, $0 < \lambda_1 < \frac{\beta p}{q}$, and $\nu(B) \leq C r^{(\beta-\alpha-\frac{q}{p})q}$ with $p' = \frac{p}{p-1}$.

Then I_α is bounded from $L^{p, \frac{Q\lambda_1}{\beta p}}(X, \mu)$ to $L^{q, \frac{\lambda_2}{q}}(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$$

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When $Q = \beta$, the previous theorem implies the following corollary.

Corollary 3.5 Let (X, δ, μ) be a β -homogeneous metric measure space, ν be a measure on X ,

$0 < \lambda_1 < \frac{\beta p}{q}$, $1 < p < \frac{\beta}{\alpha}$, and $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$. Then I_α is bounded from $L^{p, \frac{\lambda_1}{p}}(X, \mu)$ to

$L^{q, \frac{\lambda_2}{q}}(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\nu(B) \leq C \mu(B)^{q(\frac{1}{p}-\frac{\alpha}{\beta})}$$

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Corollary 3.6 Let (X, δ, μ) be a β -homogeneous metric measure space, ν be a measure on X ,

$0 < \lambda_1 < \frac{\beta p}{q}$, $1 < p < \frac{\beta}{\alpha}$, and $\nu(B) \leq C \mu(B)^{q(\frac{1}{p}-\frac{\alpha}{\beta})}$. Then I_α is bounded from

$L^{p, \frac{\lambda_1}{p}}(X, \mu)$ to $L^{q, \frac{\lambda_2}{q}}(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X ,

$$\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$$

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4. CONCLUSIONS

Through our work we have been able to extend the known results for the classical fractional integral operator I_α to the boundedness of with measure μ and ν on Morrey spaces over Q -homogeneous metric measure space. Our results not only cover the known results for I_α , but also enrich the class of funtions of α , λ_1 and λ_2 for which the operator I_α is bounded from the classical Morrey space $L^{p, \frac{Q\lambda_1}{\beta p}}(\mu)$ to $L^{q, \lambda_2}(\nu)$, on Q -homogeneous and the corollary I_α is bounded from the classical Morrey space $L^{p, \lambda_1}(\mu)$ to $L^{q, \lambda_2}(\nu)$, on β -homogeneous.

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COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper

AUTHOR'S CONTRIBUTIONS

The author read and approved the final manuscript.

ACKNOWLEDGMENTS

This research article was developed with a certain purposes related to doctorate program.

FUNDING

This research is supported by Universitas Airlangga Research and Innovation Program 2018.

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