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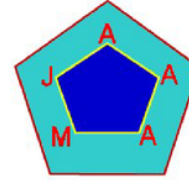
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**THE BOUNDEDNESS OF BESSEL-RIESZ OPERATORS ON GENERALIZED
MORREY SPACES**

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ABSTRACT. In this paper, we prove the boundedness of Bessel-Riesz operators on generalized Morrey spaces. The proof uses the usual dyadic decomposition, a Hedberg-type inequality for the operators, and the boundedness of Hardy-Littlewood maximal operator. Our results reveal that the norm of the operators is dominated by the norm of the kernels.

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1. INTRODUCTION

We begin with the definition of Bessel-Riesz operators. For $\gamma \geq 0$ and $0 < \alpha < n$, we define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}^{(4.2)}(x-y) f(y) dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, $p \geq 1$, where $K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^\gamma}$. Here, $I_{\alpha,\gamma}$ is called *Bessel-Riesz operator* and $K_{\alpha,\gamma}$ is called *Bessel-Riesz kernel*. The name of the kernel resembles the product of Bessel kernel and Riesz kernel [13]. While the Riesz kernel captures the local behaviour, the Bessel kernel take cares the global behaviour of the function. The Bessel-Riesz kernel is used in studying the behaviour of the solution of a Schrödinger type equation [8].

For $\gamma = 0$, we have $I_{\alpha,0} = I_\alpha$, known as *fractional integral operators* or *Riesz potentials* [12]. Around 1930, Hardy and Littlewood [5, 6] and Sobolev [11] have proved the boundedness of I_α on Lebesgue spaces via the inequality

$$\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p},$$

for every $f \in L^p(\mathbb{R}^n)$, where $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Here C denotes a constant which may depend on α, p, q , and n , but not on f .

For $1 \leq p \leq q$, the (classical) Morrey space $L^{p,q}(\mathbb{R}^n)$ is defined by

$$L^{p,q}(\mathbb{R}^n) := \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty\},$$

where $\|f\|_{L^{p,q}} := \sup_{r>0, a \in \mathbb{R}^n} r^{n(1/q-1/p)} \left(\int_{|x-a|<r} |f(x)|^p dx \right)^{1/p}$. For these spaces, we have an inclusion property which is presented by the following theorem.

Theorem 1.1. For $1 \leq p \leq q$, we have $L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n)$.

On Morrey spaces, Spanne [10] has shown that I_α is bounded from $L^{p_1,q_1}(\mathbb{R}^n)$ to $L^{p_2,q_2}(\mathbb{R}^n)$ for $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$. Furthermore, Adams [1] and Chiarenza and Frasca [2] reproved it and obtained a stronger result which is presented below.

Theorem 1.2. [Adams, Chiarenza-Frasca] If $0 < \alpha < n$, then we have

$$\|I_\alpha f\|_{L^{p_2,q_2}} \leq C \|f\|_{L^{p_1,q_1}},$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ where $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} (1 - \frac{\alpha q_1}{n})$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$.

For $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $1 \leq p < \infty$, we define the generalized Morrey space

$$L^{p,\phi}(\mathbb{R}^n) := \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\phi}} < \infty\},$$

where $\|f\|_{L^{p,\phi}} := \sup_{r>0, a \in \mathbb{R}^n} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{|x-a|<r} |f(x)|^p dx \right)^{1/p}$. Here we assume that ϕ is almost decreasing and $t^{n/p}\phi(t)$ is almost decreasing, so that ϕ satisfies the *doubling condition*, that is, there exists a constant C such that $\frac{1}{C} \leq \frac{\phi(r)}{\phi(v)} \leq C$ whenever $\frac{1}{2} \leq \frac{r}{v} \leq 2$.

In 1994, Nakai [9] obtained the boundedness of I_α from $L^{p_1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\psi}(\mathbb{R}^n)$ where $1 < p_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$ and $\int_r^\infty v^{\alpha-1}\phi(v)dv \leq Cr^\alpha\phi(r) \leq C\psi(r)$ for every $r > 0$. Nakai's result may be viewed as an extension of Spanne's. Later on, in 2009, Gunawan and Eridani [3] extended Adams-Chiarenza-Frasca's result.

Theorem 1.3. [Gunawan-Eridani] If $\int_r^\infty \frac{\phi(v)}{v} dv \leq C\phi(r)$, and $\phi(r) \leq Cr^\beta$ for every $r > 0$, $-\frac{n}{p_1} \leq \beta < -\alpha$, $1 < p_1 < \frac{n}{\alpha}$, $0 < \alpha < n$, then we have

$$\|I_\alpha f\|_{L^{p_2,\psi}} \leq C \|f\|_{L^{p_1,\phi}}$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ where $p_2 = \frac{\beta p_1}{\alpha + \beta}$ and $\psi(r) = \phi(r)^{p_1/p_2}$, $r > 0$.

The proof of the boundedness of I_α on Lebesgue spaces, Morrey spaces, or generalized Morrey spaces, usually involves Hardy-Littlewood maximal operator, which is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$ where $|B|$ denotes the Lebesgue measure of the ball $B = B(a, r)$ (centered at $a \in \mathbb{R}^n$ with radius $r > 0$). It is well known that the operator M is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$ [12, 13] and also on Morrey spaces $L^{p,q}$ for $1 < p \leq q \leq \infty$ [2].

Next, we know that $I_{\alpha,\gamma}$ is guaranteed to be bounded on generalized Morrey spaces because $K_{\alpha,\gamma}(x) \leq K_\alpha(x)$ for every $x \in \mathbb{R}^n$. The boundedness of $I_{\alpha,\gamma}$ on Lebesgue spaces can also be proved by using Young's inequality, as shown in [7].

Theorem 1.4. [7] For $\gamma > 0$ and $0 < \alpha < n$, we have $K_{\alpha,\gamma} \in L^t(\mathbb{R}^n)$ whenever $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Accordingly, we have

$$\|I_{\alpha,\gamma}f\|_{L^q} \leq \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^n)$ where $1 \leq p < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$.

Using the boundedness of Hardy-Littlewood maximal operator, we also know that $I_{\alpha,\gamma}$ is bounded on Morrey spaces.

Theorem 1.5. [7] For $\gamma > 0$ and $0 < \alpha < n$, we have

$$\|I_{\alpha,\gamma}f\|_{L^{p_2,q_2}} \leq C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,q_1}}$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ where $1 < p_1 < q_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{p_2} + 1 = \frac{1}{p_1} + \frac{1}{t}$, and $\frac{1}{q_2} + 1 = \frac{1}{q_1} + \frac{1}{t}$.

In 1999, Kurata *et al.* [8] proved the boundedness of $W \cdot I_{\alpha,\gamma}$ on generalized Morrey spaces where W is a multiplication operator. A similar result to Kurata's can be found in [3]. In the next section, we shall reprove the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on these spaces.

Theorem 1.6. (Nakai) For $1 < p \leq \infty$, we have

$$\|Mf\|_{L^{p,\phi}} \leq C \|f\|_{L^{p,\phi}},$$

for every $f \in L^{p,\phi}(\mathbb{R}^n)$.

Our results show that the norm of Bessel-Riesz operators is dominated by the norm of their kernels on (generalized) Morrey spaces.

2. MAIN RESULTS

For $\gamma > 0$ and $0 < \alpha < n$, one may observe that the kernel $K_{\alpha,\gamma}$ belongs to Lebesgue spaces $L^t(\mathbb{R}^n)$ whenever $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, where

$$\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \sim \|K_{\alpha,\gamma}\|_{L^t}^t$$

(see [7]). With this in mind, we obtain the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces as in the following theorem.

Theorem 2.1. Let $\gamma > 0$ and $0 < \alpha < n$. If $\phi(r) \leq Cr^\beta$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, and $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, then we have

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}},$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ where $p_2 = \frac{\beta p_1}{\alpha+\beta}$, and $\psi(r) = \phi(r)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\phi(r) \leq Cr^\beta$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Take $f \in L^{p_1,\phi}(\mathbb{R}^n)$ and write

$$I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x),$$

for every $x \in \mathbb{R}^n$ where $I_1(x) := \int_{|x-y| < R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^\gamma} dy$ and $I_2(x) := \int_{|x-y| \geq R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^\gamma} dy$, $R > 0$.

Using dyadic decomposition, we have the following estimate for I_1 :

$$\begin{aligned} |I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^\gamma} dy \\ &\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \\ &\leq C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/t} (2^k R)^{n/t'}}{(1+2^k R)^\gamma}. \end{aligned}$$

We then use Hölder's inequality to get

$$|I_1(x)| \leq C_2 Mf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \right)^{1/t} \left(\sum_{k=-\infty}^{-1} (2^k R)^n \right)^{1/t'}.$$

Because we have

$$\left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \right)^{1/t} \leq \left(\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \right)^{1/t} \sim \|K_{\alpha,\gamma}\|_{L^t},$$

we obtain $|I_1(x)| \leq C_3 \|K_{\alpha,\gamma}\|_{L^t} Mf(x) R^{n/t'}$.

To estimate I_2 , we use Hölder's inequality again:

$$\begin{aligned} |I_2(x)| &\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \\ &\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} (2^k R)^{n/p_1'} \left(\int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}. \end{aligned}$$

It follows that

$$\begin{aligned} |I_2(x)| &\leq C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/t}}{(1+2^k R)^\gamma} \phi(2^k R) (2^k R)^{n/t'} \\ &\leq C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/t}}{(1+2^k R)^\gamma} (2^k R)^{\beta+n/t'}. \end{aligned}$$

Another use of Hölder's inequality gives

$$|I_2(x)| \leq C_6 \|f\|_{L^{p_1, \phi}} \left(\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \right)^{1/t} \left(\sum_{k=0}^{\infty} (2^k R)^{\beta t+n} \right)^{1/t'}.$$

Because $\beta t' + n < 0$ and $\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \lesssim \|K_{\alpha, \gamma}\|_{L^t}^t$, we obtain

$$|I_2(x)| \leq C_7 \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}} R^{\beta} R^{n/t'}.$$

Summing the two estimates, we obtain

$$|I_{\alpha, \gamma} f(x)| \leq C_8 \|K_{\alpha, \gamma}\|_{L^t} \left(Mf(x) R^{n/t'} + \|f\|_{L^{p_1, \phi}} R^{n/t'+\beta} \right),$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^{\beta} = \frac{Mf(x)}{\|f\|_{L^{p_1, \phi}}}$. Hence we get a Hedberg-type inequality for $I_{\alpha, \gamma} f$, namely

$$|I_{\alpha, \gamma} f(x)| \leq C_9 \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}}^{-\alpha/\beta} Mf(x)^{1+\alpha/\beta}.$$

Now put $p_2 := \frac{\beta p_1}{\alpha + \beta}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha, \gamma} f(x)|^{p_2} dx \right)^{1/p_2} \leq C_9 \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{1/p_2}.$$

We divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ to get

$$\frac{\left(\int_{|x-a|<r} |I_{\alpha, \gamma} f(x)|^{p_2} dx \right)^{1/p_2}}{\psi(r)^{p_1/p_2}} \leq C_9 \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}}^{1-p_1/p_2} \frac{\left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{(1/p_2)}}{\phi(r)^{p_1/p_2} r^{n/p_2}},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. Taking the supremum over $a \in \mathbb{R}^n$ and $r > 0$, we obtain

$$\|I_{\alpha, \gamma} f\|_{L^{p_2, \psi}} \leq C_{10} \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1, \phi}}^{p_1/p_2}.$$

By the boundedness of the maximal operator on generalized Morrey spaces (Nakai's Theorem), the desired result follows: $\|I_{\alpha, \gamma} f\|_{L^{p_2, \psi}} \leq C \|K_{\alpha, \gamma}\|_{L^t} \|f\|_{L^{p_1, \phi}}$. ■

We note that from the inclusion property of Morrey spaces, we have

$$\|K_{\alpha, \gamma}\|_{L^{s, t}} \leq \|K_{\alpha, \gamma}\|_{L^{t, t}} = \|K_{\alpha, \gamma}\|_{L^t}$$

whenever $1 \leq s \leq t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Now we wish to obtain a more general result for the boundedness of $I_{\alpha, \gamma}$ by using the fact that the kernel $K_{\alpha, \gamma}$ belongs to Morrey spaces.

Theorem 2.2. Let $\gamma > 0$ and $0 < \alpha < n$. If $\phi(r) \leq Cr^{\beta}$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, then we have

$$\|I_{\alpha, \gamma} f\|_{L^{p_2, \psi}} \leq C \|K_{\alpha, \gamma}\|_{L^{s, t}} \|f\|_{L^{p_1, \phi}},$$

for every $f \in L^{p_1, \phi}(\mathbb{R}^n)$ where $1 \leq s \leq t$, $p_2 = \frac{\beta p_1}{\alpha + \beta}$, and $\psi(v) = \phi(v)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\phi(r) \leq C\gamma^\beta$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. As in the proof of Theorem 2.1, we have $I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$. Now, we estimate I_1 using dyadic decomposition as follow:

$$\begin{aligned} |I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^\gamma} dy \\ &\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \\ &= C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/s} (2^k R)^{n/s'}}{(1+2^k R)^\gamma}, \end{aligned}$$

where $1 \leq s \leq t$. By Hölder's inequality,

$$|I_1(x)| \leq C_2 Mf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-\infty}^{-1} (2^k R)^n \right)^{1/s'}.$$

We also have $\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \lesssim \int_{0 < |x| < R} K_{\alpha,\gamma}^s(x) dx$, so that

$$|I_1(x)| \leq C_3 Mf(x) \left(\int_{0 < |x| < R} K_{\alpha,\gamma}^s(x) dx \right)^{\frac{1}{s}} R^{n/s'} \leq C_3 \|K_{\alpha,\gamma}\|_{L^{s,t}} Mf(x) R^{n/t'}.$$

Next, we estimate I_2 by using Hölder's inequality. As in the proof of Theorem 2.1, we obtain

$$|I_2(x)| \leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} (2^k R)^{n/p_1'} \left(\int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}.$$

It thus follows that

$$\begin{aligned} |I_2(x)| &\leq C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R)}{(1+2^k R)^\gamma} \frac{\left(\int_{2^k R \leq |x-y| < 2^{k+1} R} dy \right)^{1/s}}{(2^k R)^{n/s}} \\ &\leq C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \phi(2^k R) (2^k R)^{n/t'} \frac{\left(\int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s(x-y) dy \right)^{1/s}}{(2^k R)^{n/s-n/t}} \end{aligned}$$

Because $\phi(r) \leq C\gamma^\beta$ and $\frac{\left(\int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s(x-y) dy \right)^{1/s}}{(2^k R)^{n/s-n/t}} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,t}}$ for every $k = 0, 1, 2, \dots$, we get

$$\begin{aligned} |I_2(x)| &\leq C_7 \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^k R)^{\beta+n/t'} \\ &\leq C_8 \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} R^\beta R^{n/t'}. \end{aligned}$$

From the two estimates, we obtain

$$|I_{\alpha,\gamma}f(x)| \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,t}} \left(Mf(x) R^{n/t'} + \|f\|_{L^{p_1,\phi}} R^{n/t'+\beta} \right),$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$. Hence we get

$$|I_{\alpha,\gamma}f(x)| \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{-\alpha/\beta} Mf(x)^{1+\alpha/\beta}.$$

Put $p_2 := \frac{\beta p_1}{\alpha + \beta}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2} \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{1/p_2}.$$

Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{10} \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,\phi}}^{p_1/p_2},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. With the boundedness of the maximal operator on generalized Morrey spaces (Nakai's Theorem), we obtain

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}},$$

as desired. ■

Note that by Theorem 2.2 and the inclusion of Morrey spaces, we recover Theorem 2.1:

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \leq C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}.$$

We still wish to obtain a better estimate. The following lemma presents that the Bessel-Riesz kernels belong to generalized Morrey space $L^{s,\sigma}(\mathbb{R}^n)$ for some $s \geq 1$ and a suitable function σ .

Lemma 2.3. Suppose that $\gamma > 0$ and $0 < \alpha < n$. If $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies

$$\int_{0 < r \leq R} r^{(\alpha-n)s+n-1} dr \leq C \sigma^s(R) R^n$$

for every $R > 0$, then $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$.

Proof. Suppose that the hypothesis holds. It is sufficient to evaluate the integral around 0. We observe that

$$\int_{|x| \leq R} K_{\alpha,\gamma}^s(x) dx = \int_{|x| \leq R} \frac{|x|^{(\alpha-n)s}}{(1+|x|)^{\gamma s}} dx \leq C \int_{0 < r \leq R} r^{(\alpha-n)s+n-1} dr \leq C \sigma^s(R) R^n.$$

We divide both sides of the inequality by $\sigma^s(R) R^n$ and take s^{th} -root to obtain

$$\frac{\left(\int_{|x| \leq R} K_{\alpha,\gamma}^s(x) dx \right)^{1/s}}{\sigma(R) R^{n/s}} \leq C^{1/s}.$$

Now, taking the supremum over $R > 0$, we have

$$\sup_{R > 0} \frac{\left(\int_{|x| \leq R} K_{\alpha,\gamma}^s(x) dx \right)^{1/s}}{\sigma(R) R^{n/s}} < \infty.$$

Hence $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$. ■

By the hypothesis of Lemma 2.3 we also obtain $\frac{\left(\int_{2^k R < |x| \leq 2^{k+1} R} K_{\alpha,\gamma}^s(x) dx \right)^{1/s}}{\sigma(2^k R) (2^k R)^{n/s}} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,\sigma}}$ for every integer k and $R > 0$. Moreover, $\frac{\left(\sum_{k=-1}^{\infty} K_{\alpha,\gamma}^s(2^k R) (2^k R)^n \right)^{1/s}}{\sigma(R) (R)^{n/s}} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,\sigma}}$ holds for every $R > 0$. One may observe that $1 \leq s \leq \frac{n \ln R_1}{-\ln \sigma(R_1)}$ for every $R_1 > 1$. For $\sigma(R) = R^{-n/t}$, this inequality reduces to $1 \leq s \leq t$.

We shall now use the lemma to prove the following theorem.

Theorem 2.4. Suppose that $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the doubling condition and $\sigma(r) \leq Cr^{-\alpha}$ for every $r > 0$, so that $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$ for $1 \leq s < \frac{n}{n-\alpha}$, where $0 < \alpha < n$ and $\gamma > 0$. If $\phi(r) \leq Cr^\beta$ for every $r > 0$, where $-\frac{n}{p_1} < \beta < -\alpha$, then we have

$$\|I_{\alpha,\gamma}f\|_{L^{p_2,\psi}} \leq C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$, where $1 < p_1 < \frac{n}{\alpha}$, $p_2 = \frac{\beta p_1}{\beta+n-\alpha}$ and $\psi(r) = \phi(r)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the doubling condition and $\sigma(r) \leq Cr^{-\alpha}$ for every $r > 0$, such that $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$ for $1 \leq s < \frac{n}{n-\alpha}$. Suppose also that $\phi(r) \leq Cr^\beta$ for every $r > 0$, where $-\frac{n}{p_1} < \beta < -\alpha$, $1 < p_1 < \frac{n}{\alpha}$. As in the proof of Theorem 2.1, we write $I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$. As usual, we estimate I_1 by using dyadic decomposition:

$$\begin{aligned} |I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^\gamma} dy \\ &\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \\ &= C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/s} (2^k R)^{n/s'}}{(1+2^k R)^\gamma} \end{aligned}$$

By using Hölder inequality, we obtain

$$|I_1(x)| \leq C_2 Mf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-\infty}^{-1} (2^k R)^n \right)^{1/s'}$$

But $\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \lesssim \int_{0 < |x| < R} K_{\alpha,\gamma}^s(x) dx$, and so we get

$$\begin{aligned} |I_1(x)| &\leq C_2 Mf(x) \left(\int_{0 < |x| < R} K_{\alpha,\gamma}^s(x) dx \right)^{\frac{1}{s}} R^{n/s'} \\ &\leq C_2 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} Mf(x) \sigma(R) R^n \\ &\leq C_3 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} Mf(x) R^{n-\alpha}. \end{aligned}$$

Next, we estimate I_2 as follows:

$$\begin{aligned} |I_2(x)| &\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \\ &\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} (2^k R)^{n/p_1} \left(\int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1} \\ &\leq C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R) (2^k R)^n \left(\int_{2^k R \leq |x-y| < 2^{k+1} R} dy \right)^{1/s}}{(1+2^k R)^\gamma (2^k R)^{n/s}} \\ &\leq C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^k R)^{n-\alpha+\beta} \frac{\left(\int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s(x-y) dy \right)^{1/s}}{\sigma(2^k R) (2^k R)^{n/s}} \end{aligned}$$

Because $\frac{\int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s(x-y) dy}{\sigma(2^k R)(2^k R)^n} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,\sigma}}^s$ for every $k = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} |I_2(x)| &\leq C_6 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^k R)^{n-\alpha+\beta} \\ &\leq C_7 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta}. \end{aligned}$$

31 It follows from the above estimates for I_1 and I_2 that

$$|I_{\alpha,\gamma} f(x)| \leq C_8 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} (Mf(x) R^{n-\alpha} + \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta})$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$, whence

$$|I_{\alpha,\gamma} f(x)| \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{(\alpha-n)/\beta} Mf(x)^{1+(n-\alpha)/\beta}.$$

Put $p_2 := \frac{\beta p_1}{\beta+n-\alpha}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2} \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{(1/p_2)}.$$

Divide the both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ to get

$$\begin{aligned} \frac{\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2}}{\psi(r) r^{n/p_2}} &= \frac{\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2}}{\phi(r)^{p_1/p_2} r^{n/p_2}} \\ &\leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \frac{\left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{(1/p_2)}}{\phi(r)^{p_1/p_2} r^{n/p_2}}, \end{aligned}$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. Finally, take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to obtain

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{10} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,\phi}}^{p_1/p_2}.$$

30 Because the maximal operator is bounded on generalized Morrey spaces (Nakai's Theorem), we conclude that $\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}$. ■

3. CONCLUDING REMARKS

4 The results presented in this paper, namely Theorems 2.1, 2.2, and 2.4, extend the results on the boundedness of Bessel-Riesz operators on Morrey spaces [7]. Similar to Gunawan-Eridani's result for I_α , Theorems 2.1, 2.2, and 2.4 ensures that $I_{\alpha,\gamma} : L^{p_1,\phi}(\mathbb{R}^n) \rightarrow L^{p_2,\phi^{p_1/p_2}}(\mathbb{R}^n)$. Notice that if we have $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for $t \in (\frac{n}{n+\gamma-\alpha}, \frac{n}{n-\alpha})$, then $R^{-n/t} < \sigma(R)$ holds for every $R > 0$, then Theorem 2.4 gives a better estimate than Theorem 2.2. Now, if we define $\sigma(R) := (1 + R^{n/t_1}) R^{-n/t}$ for some $t_1 > t$, then $\|K_{\alpha,\gamma}\|_{L^{s,\sigma}} < \|K_{\alpha,\gamma}\|_{L^{s,t}}$. By Theorem 2.2 and the inclusion property of Morrey spaces, we obtain

$$\begin{aligned} \|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} &\leq C \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \\ &< C \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \\ &\leq C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}. \end{aligned}$$

We can therefore say that Theorem 2.4 gives the best estimate among the three. Furthermore, we have also shown that, in each theorem, the norm of Bessel-Riesz operators on generalized Morrey spaces is dominated by that of Bessel-Riesz kernels.

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