The Boundedness of Bessel-Riesz Operators On Morrey Spaces

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Abstract. In this paper, we shall discuss about Bessel-Riesz operators. Kurata *et al.* have investigated their boundedness on generalized Morrey spaces with weight. The boundedness of these operators on Lebesgue spaces and Morrey spaces will be reproved using a different approach. Moreover, we also find the norm of the operators are bounded by the norm of the kernels.

Keywords: Bessel-Riesz operators, Hardy-Littlewood maximal operator, Morrey spaces.

INTRODUCTION

Let $0 < \gamma$, $0 < \alpha < n$ and define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y) f(y)\, dy$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, where $p \geq 1$, $K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^{\gamma}}$, $x \in \mathbb{R}^n$. Here, $K_{\alpha,\gamma}$ can be viewed as multiple of two kernels, $K_{\alpha,\gamma}(x) = J_\gamma(x) K_\alpha(x)$ for every $x \in \mathbb{R}^n$. In [1], J_γ and K_α are known as *Bessel kernel* and *Riesz kernel*. So, $K_{\alpha,\gamma}$ is called *Bessel-Riesz kernel* and $I_{\alpha,\gamma}$ is called *Bessel-Riesz operator*.

For $\gamma = 0$, we have $I_{\alpha,0} = I_\alpha$ (is called *fractional integral operator* or *Riesz potential* [1]). Studies about I_α were started since 1920's. Hardy-Littlewood [2, 3] and Sobolev [4] proved the boundedness of I_α on *Lebesgue spaces* through the inequality $\|I_\alpha f\|_{L^q} \leq C_p \|f\|_{L^p}$, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \frac{\alpha}{n}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

For $1 \leq p \leq q$, *the (classical) Morrey space* $L^{p,q}(\mathbb{R}^n)$ is defined by

$$L^{p,q}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty \right\},$$

where $\|f\|_{L^{p,q}} := \sup_{r>0, a\in\mathbb{R}^n} r^{n(1/q-1/p)} \left(\int_{|x-a|<r} |f(x)|^p\, dx \right)^{1/p}$. We have an inclusion property for Morrey spaces $L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n)$.

On Morrey spaces, Spanne [5] has shown that I_α is bounded form $L^{p_1,q_1}(\mathbb{R}^n)$ to $L^{p_2,q_2}(\mathbb{R}^n)$ for $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$. Furthermore, Adams [6] and Chiarenza-Frasca [7] obtained a stronger result.

Theorem 1 *[Adams, Chiarenza-Frasca] If* $0 < \alpha < n$ *then we have*

$$\|I_\alpha f\|_{L^{p_2,q_2}} \leq C_{p_1,q_1} \|f\|_{L^{p_1,q_1}} ,$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ *where* $1 < p_1 < q_1 < \frac{n}{\alpha}, \frac{1}{p_2} = \frac{1}{p_1}\left(1 - \frac{\alpha q_1}{n}\right)$, *and* $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$.

Meanwhile, we have $\left|I_{\alpha,\gamma} f(x)\right| \leq |I_\alpha f(x)|$, for every $f \in L^p_{loc}(\mathbb{R}^n)$. Using this inequality, $I_{\alpha,\gamma}$ is bounded on these spaces. In 1999, Kurata *et. al* [8] have proved that $W \cdot I_{\alpha,\gamma}$ is bounded on generalized Morrey spaces where W is a multiplication operator. Here, we shall discuss about the boundedness of $I_{\alpha,\gamma}$ on Lebesgue spaces and Morrey spaces. We shall see the influence of $K_{\alpha,\gamma}$ for the boundedness of $I_{\alpha,\gamma}$.

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