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FRACTIONAL INTEGRAL OPERATORS IN GENERALIZED MORREY SPACES DEFINED ON METRIC MEASURE SPACES

ERIDANI AND Y. SAWANO

Abstract. We derive some necessary and sufficient conditions for the boundedness of fractional integral operators in generalized Morrey spaces defined on metric measure spaces.

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1. Introduction

In the present paper we consider the boundedness of the fractional integral operators on metric measure spaces (X, ρ, μ). By this we mean that (X, ρ) is a metric space and μ is a Borel measure. By generalizing the underlying measures, we seek for a better understanding of the fractional integral operators. It seems that Morrey spaces can describe the boundedness property of fractional integral operators very precisely. The most fundamental result of this field is due to Adams [1]. Nowadays there are series of papers that describe the boundedness property of fractional integral operators by means of (generalized) Morrey spaces (see for example, [5, 4, 7, 10, 15, 17]). The boundedness of fractional integral operators defined on nonhomogeneous spaces on $\mathbb{R}^n$ was established in [8] and the same problem on general nonhomogeneous spaces was investigated in [9]. A remarkable progress on function spaces on metric measure spaces was made a decade ago, starting from the papers [11, 18, 19].

To describe our setting, we need some notations. Denote by $\mathcal{B}(X)$ the set of all open balls in X. Throughout the present paper we postulate the following conditions on ϕ: Here and below we denote by $B(a, r)$ the open ball centered at a and of radius $r > 0$. For a ball $B := B(a, r)$, we sometimes write $\phi(a, r) := \phi(B)$. In what follows the letter C will be used to denote constants that may change from one occurrence to another one.

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($\phi 1$) A set function $\phi : \mathcal{B}(X) \to [0,\infty)$ is almost decreasing. Namely, there exists a constant $C > 0$ such that $\phi(B_2) \le C\,\phi(B_1)$ for all balls B_1 and B_2 with $B_1 \subseteq B_2$.

($\phi 2p$) Let $1 \le p < \infty$. The function ϕ and the measure μ are related as follows: there exists a constant $C > 0$ such that $\phi(B_1)^p \mu(B_1) \le C\,\phi(B_2)^p \mu(B_2)$ for all pairs of balls B_1, B_2 such that $B_1 \subseteq B_2$.

As a direct consequence of ($\phi 1$), there exists a constant $C > 0$ with the following properties:

$$C^{-1}\phi(a,2r) \le \phi(a,t) \le C\phi(a,r),$$
$$C^{-1}\,\frac{\phi(a,2r)}{r} \le \frac{\phi(a,t)}{t} \le C\,\frac{\phi(a,r)}{r},$$
$$C^{-1}\,\phi(a,2r) \le \int_r^{2r} \frac{\phi(a,t)}{t}\,dt \le C\,\phi(a,r)$$

for all $0 < r < t < 2r$ and $a \in X$.

In the present paper we place ourselves in the setting of generalized Morrey spaces on homegenous or nonhomogeneous spaces.

We say that $X := (X,\rho,\mu)$ is a *homogeneous* metric measure space if μ satisfies *the doubling property.* That is, there exists a constant $C > 0$ such that for every balls $B := B(a,r)$,

$$(D\mu) \quad \mu(B(a,2r)) \le C\,\mu(B(a,r)).$$

Otherwise, $X := (X,\rho,\mu)$ is said to be a *nonhomogeneous* space.

If we are given a function $\phi : \mathcal{B}(X) \to [0,\infty)$, we define the generalized Morrey space $L^p_\phi(\nu,\mu)$ as the set $f \in L^p_{\rm loc}(\nu)$ satisfying

$$\left\| f : L^p_\phi(\nu,\mu) \right\| := \sup_{B\in\mathcal{B}(X)} \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(y)|^p d\nu(y) \right)^{1/p} < \infty.$$

The measures μ and ν are necessary for the definition in order to cover plausible weighted settings. If $\mu = \nu$, then we abbreviate $L^p_\phi(\nu,\mu)$ to $L^p_\phi(\mu)$. As a starting point we prove the theorem, ensuring that $L^p_\phi(\mu)$ is not empty.

Proposition A. *We write $B_0 := B(a_0,r_0)$. If μ and ϕ satisfy ($\phi 1$) and ($D\mu$) respectively, then we have*

$$\frac{1}{\phi(B_0)} \le \left\| \chi_{B_0} : L^p_\phi(\mu) \right\| \le \frac{C}{\phi(B_0)}$$

for some universal constant $C > 1$.

Generalized Morrey spaces are nowadays not for the sake of generalization, but for its own sake. They come naturally into play for potential theory. The classical Morrey space $\mathcal{M}_{p,q}(\mathbb{R}^n)$ with $1 < q \le p < \infty$ is defined as the set of measurable functions endowed with the norm

$$\|f\|_{\mathcal{M}_{p,q}} := \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q dx \right)^{\frac{1}{q}}, \tag{1}$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of dyadic cubes in $\mathbb{R}^n$. Let $1 < q < p < \infty$. Then there exists a positive constant $C_{p,q}$ such that

$$\int_Q |f(x)|dx \le C_{p,q}|Q|(1+|Q|)^{-\frac{1}{p}} \log\left(e + \frac{1}{|Q|}\right) \left\|(1-\Delta)^{\frac{n}{2p}} f\right\|_{\mathcal{M}_{p,q}}$$

holds for all $f \in \mathcal{M}_{p,q}(\mathbb{R}^n)$ and for all cubes $Q \in \mathcal{D}(\mathbb{R}^n)$.

Let $0 < r < \infty$ and $\Phi : [0,\infty) \to [0,\infty)$ be a suitable function. For a function f, locally in $L_r(\mathbb{R}^n)$, we set

$$\|f\|_{\mathcal{M}_{\Phi,r}} := \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} \Phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(x)|^r \, dx \right)^{\frac{1}{r}},$$

where $\ell(Q)$ denotes the side-length of the cube Q. Thus in words of this generalized Morrey norm, by letting

$$\Phi(t) = t^n(1+t^n)^{-\frac{1}{p}} \left(\log\left(e + \frac{1}{t^n}\right) \right)^{-1} \quad \text{for } t \in [0,\infty),$$

and taking (1) into account, we have

$$\|f\|_{\mathcal{M}_{\Phi,1}} \le C_{p,q} \left\|(1-\Delta)^{\frac{n}{2p}} f\right\|_{\mathcal{M}_{p,q}}.$$

See [16] for details.

This paper is organized as follows: We place ourselves in the different settings in each section. In Section 2 we investigate the function spaces endowed with a doubling Radon measure and investigate the boundedness of fractional integral operators in generalized Morrey spaces. In Section 3 we consider the fractional maximal operator on a metric measure space with a doubling Radon measure. Finally, in Section 4 we place ourselves in the setting of a metric measure space with a general Radon measure satisfying the growth condition. Our result is concerned not with the one in [1] but with the one of the paper due to Spanne. Note that the result due to Spanne is contained in [12].

2. Morrey spaces on Homogeneous Spaces

In this section we prove Proposition A and discuss fractional integral operators in Morrey spaces on homogeneous spaces (X, ρ, μ).

Proof of Proposition A. It follows immediately from the definition that

$$\|\chi_{B_0} : L^p_\phi(\mu)\| = \sup_{B\in\mathcal{B}(X)} \frac{1}{\phi(B)}\left(\frac{\mu(B\cap B_0)}{\mu(B)}\right)^{1/p} =$$

$$= \sup_{B\in\mathcal{B}(X)} \frac{1}{\phi(B)}\left(\frac{\mu(B\cap B(a_0,r_0))}{\mu(B)}\right)^{1/p}.$$

Although B in the sup above runs over all the balls, we do not have to take B into account unless $B\cap B_0 \neq \emptyset$. Keeping this in mind, we let $B := B(a,r)$ be such a ball. If $r \le r_0$, then a geometric observation shows $B(a,r) \subseteq B(a_0, 3r_0)$. Consequently, by the doubling property of μ,

$$\mu(B(a,r)) \le \mu(B(a_0,3r_0)) \le \mu(B(a_0,4r_0)) \le C\mu(B(a_0,r_0))$$

and

$$\mu(B(a_0,3r_0)) \ge \mu(B_0).$$

So, by ($\phi1$) and ($\phi2p$) together with the doubling property of μ, we have

$$\frac{1}{\phi(B)}\left(\frac{\mu(B\cap B_0)}{\mu(B)}\right)^{1/p} \le \frac{1}{\phi(B)} \le \frac{C}{\phi(B(a_0,3r_0))} \le \frac{C}{\phi(B_0)}. \tag{2}$$

Suppose now that $r_0 < r$. Then we have $B_0 \subset 3B$ and

$$\mu(3B) \le \mu(4B) \le C\mu(B).$$

Consequently, by virtue of ($\phi2p$) we have

$$\frac{1}{\phi(B)}\left(\frac{\mu(B\cap B_0)}{\mu(B)}\right)^{1/p} \le \frac{1}{\phi(B)}\left(\frac{\mu(B_0)}{\mu(B)}\right)^{1/p} \le \frac{C}{\phi(B)}\left(\frac{\mu(B_0)}{\mu(3B)}\right)^{1/p} \le$$

$$\le \frac{C}{\phi(B_0)}. \tag{3}$$

Inequalities (2) and (3) yield the upper bound of $\|\chi_{B_0} : L^p_\phi(\mu)\|$.

Meanwhile, if we let $B = B_0$, then we obtain the left-hand side inequality. □

Consider, for $0 < \alpha < 1$, the following fractional integral operator

$$K_\alpha f(x) := \int_X f(y)\mu\big(B(x,\rho(x,y))\big)^{\alpha-1}\, d\mu(y).$$

For the related definitions of this type of operators, we refer to [13, 14]. In particular, the following theorem holds (see [3, Theorem 6.2.1]).

Theorem A. *Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Let μ and ν be Radon measures on X. Then K_α is bounded from $L^p(X,\mu)$ to $L^q(X,\nu)$ if and only if there exists $C > 0$ such that*

$$\nu(B) \le C\mu(B)^{q(1/p-\alpha)}$$

for all balls B.

In analogy with Theorem A, we prove the following result below.

Theorem B. *Let $1 < p < q < \infty$ and $\alpha \in (0, 1/p)$. Assume in addition that: $1/p - 1/q = \alpha$, that ϕ fulfills $(\phi 1)$ and $(\phi 2p)$ and there exists a constant $C > 0$ such that*

$$C^{-1}\psi(B) \le \mu(B)^{-1/q+1/p}\phi(B) \le C\psi(B) \tag{4}$$

and

$$\int_r^\infty \mu(B(a,t))^\alpha \frac{\phi(a,t)}{t}\, dt \le C\mu(B(a,r))^\alpha \phi(a,r), \quad a \in X, \quad r > 0, \tag{5}$$

then the necessary and sufficient condition for the boundedness of K_α from $L^p_\phi(\mu)$ to $L^q_\psi(\nu, \mu)$ is

$$\nu(B) \le C\mu(B) \quad \text{for all} \quad B \in \mathcal{B}(X)$$

for some constant $C > 0$.

Remark. Theorem B can be considered as a generalization of [3, Theorem 3.1] in the special case when ρ is a metric, $1/p - 1/q = \alpha$, $\phi(B) = \mu(B)^{(\lambda_1 - 1)/p}$, $\psi(B) = \mu(B)^{(\lambda_2 - 1)/q}$, where $0 < \lambda_1 < 1 - \alpha p$, $\lambda_2/q = \lambda_1/p$.

Proof. Sufficiency. Let $f \in L^p_\phi(\mu)$. Fix a ball $B = B(a, r)$ in X. Denote by $\tilde{B}$ the double of B; $\tilde{B} = B(a, 2r)$. We decompose

$$f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^{\mathrm{C}}}. \tag{6}$$

From the definition of the Morrey norm $\|\cdot : L^p_\phi(\mu)\|$, we have $f_1 \in L^p(\mu)$. More quantitatively, we have

$$\big\|f_1 : L^p(\mu)\big\| \le \mu(B)^{1/p}\phi(a,r)\big\|f : L^p_\phi(\mu)\big\| < \infty. \tag{7}$$

If we invoke Theorem A,

$$\left(\frac{1}{\mu(B)}\int_B \big|K_\alpha f_1(x)\big|^q\, d\nu(x)\right)^{1/q} \le \mu(B)^{-1/q}\big\|K_\alpha f_1 : L^q(\nu)\big\| \le$$
$$\le \mu(B)^{-1/q}\|K_\alpha\|_{L^p(\mu)\to L^q(\nu)}\big\|f_1 : L^p(\mu)\big\|.$$

By using (7), we obtain

$$\left(\frac{1}{\mu(B)}\int_B \big|K_\alpha f_1(x)\big|^q\, d\nu(x)\right)^{1/q} \le$$
$$\le \|K_\alpha\|_{L^p(\mu)\to L^q(\nu)}\mu(B)^{1/p-1/q}\phi(B)\,\big\|f : L^p_\phi(\mu)\big\|.$$

Finally, by virtue of (4), it follows that

$$\left(\frac{1}{\mu(B)}\int_B \left|K_\alpha f_1(x)\right|^q \, d\nu(x)\right)^{1/q} \le C\|K_\alpha\|_{L^p(\mu)\to L^q(\nu)}\psi(B)\,\left\|f : L^p_\phi(\mu)\right\|.$$

Thus, the estimate of $K_\alpha f_1$ is valid, and now we have

$$\frac{1}{\psi(B)}\left(\frac{1}{\mu(B)}\int_B \left|K_\alpha f_1(x)\right|^q \, d\nu(x)\right)^{1/q} \le$$
$$\le C\|K_\alpha\|_{L^p(\mu)\to L^q(\nu)}\,\left\|f : L^p_\phi(\mu)\right\|. \qquad (8)$$

Now we estimate $K_\alpha f_2$. We proceed as in [6]. For each $t \in B = B(a,r)$, we have uniform over t estimate

$$\left|K_\alpha f_2(t)\right| \le \sum_{k=1}^{\infty} \int_{2^k r\le\rho(t,y)<2^{k+1}r} \frac{|f(y)|}{\mu(B(t,\rho(t,y)))^{1-\alpha}}\, d\mu(y).$$

On each integral domain $2^k r \le \rho(t,y) < 2^{k+1} r$ of t, we find

$$\left|K_\alpha f_2(t)\right| \le \left\|f : L^p_\phi(\mu)\right\| \sum_{k=1}^{\infty} \mu(B(t,2^k r))^{\alpha-1}\mu(B(a,2^{k+1}r))\phi(a,2^{k+1}r).$$

By the doubling property of μ, we have

$$\left|K_\alpha f_2(t)\right| \le C\left\|f : L^p_\phi(\mu)\right\| \sum_{k=1}^{\infty} \mu(B(t,2^k r))^{\alpha}\phi(a,2^{k+1}r).$$

Taking now into account that $\displaystyle\int_b^{2b}\frac{dt}{t} = \log 2 \quad (b>0)$ and (5), we have

$$\left|K_\alpha f_2(t)\right| \le C\left\|f : L^p_\phi(\mu)\right\| \int_r^{\infty} \mu(B(a,s))^{\alpha}\frac{\phi(a,s)}{s}\, ds \le$$
$$\le C\left\|f : L^p_\phi(\mu)\right\|\mu(B(a,r))^{\alpha}\phi(a,r).$$

So, for every ball B, by virtue of the assumption $1/q = 1/p - \alpha$, we derive

$$\left(\frac{1}{\mu(B)}\int_B \left|K_\alpha f_2(x)\right|^q \, d\nu(x)\right)^{1/q} \le C\left\|f : L^p_\phi(\mu)\right\|\mu(B)^{\alpha}\phi(B) \le$$
$$\le C\left\|f : L^p_\phi(\mu)\right\|\psi(B).$$

Consequently, we obtain

$$\frac{1}{\psi(B)}\left(\frac{1}{\mu(B)}\int_B \left|K_\alpha f_2(x)\right|^q d\nu(x)\right)^{1/q} \leq$$
$$\leq C\|K_\alpha\|_{L^p(\mu)\to L^q(\nu)} \left\|f : L^p_\phi(\mu)\right\|. \tag{9}$$

If we put (8) and (9) together, we will have

$$\frac{1}{\psi(B)}\left(\frac{1}{\mu(B)}\int_B \left|K_\alpha f(x)\right|^q d\nu(x)\right)^{1/q} \leq C\|K_\alpha\|_{L^p(\mu)\to L^q(\nu)} \left\|f : L^p_\phi(\mu)\right\|.$$

Thus, it follows that K_α is bounded from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$.

Necessity. Assume instead that K_α is bounded from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$. Our current testing condition is

$$\left\|K_\alpha \chi_{B_0}\right\|_{L^q_\psi(\nu,\mu)} \leq \|K_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)} \|\chi_{B_0}\|_{L^p_\phi(\mu)}. \tag{10}$$

From the definition of the integral operator K_α, we have

$$K_\alpha \chi_{B_0}(x) = \int_{B_0} \mu\big(B(x,\rho(x,y))\big)^{\alpha-1} d\mu(y) \geq \int_{B_0} \mu(B(x,r_0))^{\alpha-1} d\mu(y) =$$
$$= \mu(B_0)^\alpha,$$

for all $x \in B_0 := B(a_0, r_0)$. Consequently, by the definition of the Morrey norm $\|\cdot : L^q_\psi(\nu,\mu)\|$ and (10), we find that

$$\mu(B_0)^\alpha \leq \left(\frac{1}{\nu(B_0)}\int_{B_0} \left|K_\alpha \chi_{B_0}(x)\right|^q d\nu(x)\right)^{1/q} \leq$$
$$\leq \nu(B_0)^{-1/q}\mu(B_0)^{1/q}\left\|K_\alpha\chi_{B_0} : L^q_\psi(\nu,\mu)\right\|\psi(B_0) \leq$$
$$\leq \|K_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)}\nu(B_0)^{-1/q}\mu(B_0)^{1/q}\left\|\chi_{B_0} : L^p_\phi(\mu)\right\|\psi(B_0).$$

If we use Proposition A, then we have

$$\mu(B_0)^\alpha \leq C\|K_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)}\nu(B_0)^{-1/q}\mu(B_0)^{1/q}\phi(B_0)^{-1}\psi(B_0).$$

Arranging this inequality and (4), we obtain

$$\nu(B_0) \leq (C\|K_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)})^q\mu(B_0),$$

which completes the proof of the sufficiency. □

3. Fractional Maximal Function on Homogeneous Spaces

We now consider the following (centered) fractional maximal operator

$$M_\alpha f(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))^{1-\alpha}} \int\limits_{B(x,r)} |f(y)|\, d\mu(y), \quad 0<\alpha<1.$$

For any positive measurable function $f : X \to [0,\infty]$, we have a pointwise estimate

$$M_\alpha f(x) \le K_\alpha f(x) \tag{11}$$

for some constant, independent of f.

Our aim here is to prove the following result.

Theorem C. *Let $1<p<q<\infty$ and $\alpha \in (0,1/p)$. Assume that $1/p-1/q=\alpha$, (4) and (5) hold and that ϕ fulfills ($\phi1$) and ($\phi2p$). Then the necessary and sufficient condition for the boundedness of M_α from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$ is that there exists $C>0$ such that*

$$\nu(B) \le C\mu(B) \quad \text{for all } B \in \mathcal{B}(X). \tag{12}$$

Proof. Necessity. Suppose $x \in B_0 := B(a_0, r_0)$, and M_α is bounded from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$. Directly from the definition of the fractional maximal operator, we have

$$\mu(B_0)^\alpha \le M_\alpha \chi_{B_0}(x).$$

Also, by the definition of the Morrey norm $\|\cdot : L^q_\psi(\nu,\mu)\|$, we have

$$\mu(B_0)^\alpha \le \left(\frac{1}{\nu(B_0)} \int\limits_{B_0} \left|M_\alpha \chi_{B_0}(x)\right|^q d\nu(x)\right)^{1/q} \le$$

$$\le \nu(B_0)^{-1/q} \mu(B_0)^{1/q} \left\|M_\alpha \chi_{B_0} : L^q_\psi(\nu,\mu)\right\| \psi(B_0).$$

If we use the boundedness of M_α, then we will have

$$\mu(B_0)^\alpha \le \|M_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)} \nu(B_0)^{-1/q} \mu(B_0)^{1/q} \left\|\chi_{B_0} : L^p_\phi(\mu)\right\| \psi(B_0).$$

By invoking now Proposition A, we deduce

$$\mu(B_0)^\alpha \le C\|M_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)} \nu(B_0)^{-1/q} \mu(B_0)^{1/q} \phi(B_0)^{-1} \psi(B_0).$$

Hence, by (5) we have

$$\nu(B)^{1/q} \le C\|M_\alpha\|_{L^p_\phi(\mu)\to L^q_\psi(\nu,\mu)} \mu(B)^{1/p-\alpha}.$$

Sufficiency. This is an immediate consequence of Theorem B and (11). Indeed, assuming (12), we have K_α is bounded from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$, by virtue of Theorem B.

Indeed, using (11) and the boundedness of K_α from $L^p_\phi(\mu)$ to $L^q_\psi(\nu,\mu)$ in this order, we have

$$\|M_\alpha f\|_{L^q_\psi(\nu,\mu)} \le \big\| K_\alpha[|f|] \big\|_{L^q_\psi(\nu,\mu)} \le C\|f\|_{L^p_\phi(\mu)}. \qquad \square$$

4. Nonhomogeneous Morrey Spaces

Let now $X := (X,\rho,\mu)$ be a *nonhomogeneous* measure metric space. We consider the following fractional integral operator

$$I_\alpha f(\mathbf{t}) := \int_X f(y)\rho(\mathbf{t},y)^{\alpha-1}\,d\mu(y) \quad (\mathbf{t}\in X),$$

where $0<\alpha<1$. Here and below to denote a point in X, we use $\mathbf{t}$, while t denotes as usual a positive real number.

In this space, we define the (nonhomogeneous) Morrey space $M^p_\phi(\mu;s)$ as follows;

$$f\in M^p_\phi(\mu;s) \Leftrightarrow \left\|f : M^p_\phi(\mu;s)\right\| := \sup_B \frac{1}{\phi(r)}\left(\frac{1}{r^s}\int_B |f(y)|^p\,d\mu(y)\right)^{1/p} < \infty.$$

We assume that $\phi:(0,\infty)\to(0,\infty)$ is a decreasing positive function.

The following is proved by Kokilashvili and Meskhi [9]. By García-Cuerva and Gatto [2] the case where $X=\mathbb{R}^d$ and $s=1$ was studied.

Theorem D. *Assume*

$$1<p<q<\infty, \quad 0<\alpha<1, \quad s=\frac{pq(1-\alpha)}{pq+p-q}. \tag{13}$$

Let (X,ρ,μ) be a nonhomogeneous space. Then I_α is bounded from $L^p(X)$ to $L^q(X)$, if and only if μ satisfies the growth condition

$$\mu(B(\mathbf{t},r)) \le Cr^s,$$

for all $B=B(\mathbf{t},r)\in\mathcal{B}(X)$.

Motivated by the above result, we prove the following

Theorem E. *Suppose that $1<p<q<\infty$ and $0<\alpha<1/p$. Assume*

$$s=\frac{pq(1-\alpha)}{pq+p-q}. \tag{14}$$

Assume that there exists a constant $C>0$ such that

$$\int_r^\infty t^{\alpha+s-2}\phi(t)\,dt \le Cr^{\alpha+s-1}\phi(r), \tag{15}$$

for every $r>0$. Assume, in addition, that $\psi:(0,\infty)\to(0,\infty)$ satisfies

$$C^{-1}\psi(r) \le r^{\alpha+s-1}\phi(r) \le C\psi(r) \quad (r>0) \tag{16}$$

for some positive constant C. Then the sufficient condition for the boundedness of I_α from $M^p_\phi(\mu; s)$ to $M^q_\psi(\mu; s)$ is that there exists $C > 0$ such that the growth condition

$$\mu(B(\mathbf{t}, r)) \le Cr^s$$

holds for all $B = B(\mathbf{t}, r) \in \mathcal{B}(X)$.

Note that this generalizes [6, Theorem 3.4].

Proof. Sufficiency. Let $B = B(\mathbf{t}, r) \in \mathcal{B}(X)$ be fixed and denote by $\tilde{B}$ its double; $\tilde{B} = B(\mathbf{t}, 2r)$. For every $f \in M^p_\phi(\mu)$, write

$$f = f_1 + f_2 = f\chi_{\tilde{B}} + f\chi_{\tilde{B}^C}. \tag{17}$$

The treatment of f_1 is simple. Note that $f_1 \in L^p(\mu)$. More quantitatively, we have

$$\|f_1 : L^p(\mu)\| \le \phi(r) r^{s/p} \|f : M^p_\phi(\mu; s)\| < \infty.$$

Consequently, if we invoke Theorem D, then we will have

$$\left(\frac{1}{r^s}\int_B |I_\alpha f_1(x)|^q \, d\mu(x)\right)^{1/q} \le \|I_\alpha\|_{L^p(\mu)\to L^q(\mu)} \phi(r) r^{s(1/p-1/q)} \|f : M^p_\phi(\mu; s)\| =$$

$$= \|I_\alpha\|_{L^p(\mu)\to L^q(\mu)} \phi(r) r^{s+\alpha-1} \|f : M^p_\phi(\mu; s)\|.$$

Consequently from (16), we obtain

$$\frac{1}{\psi(r)}\left(\frac{1}{r^s}\int_B |I_\alpha f_1(x)|^q \, d\mu(x)\right)^{1/q} \le C \|I_\alpha\|_{L^p(\mu)\to L^q(\mu)} \|f : M^p_\phi(\mu; s)\|. \tag{18}$$

Let us now deal with f_2. To this end we fix a point $x \in B$. Then we have

$$|I_\alpha f_2(x)| \le \int_{\tilde{B}^C} \frac{|f(y)|}{\rho(x,y)^{1-\alpha}} \, d\mu(y) \le 2^{1-\alpha} \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{1-\alpha}} \int_{\rho(x,y)<2^{k+1}r} |f(y)| \, d\mu(y).$$

In view of the definition of the Morrey norm, we have

$$|I_\alpha f_2(x)| \le C \|f : M^p_\phi(\mu; s)\| \sum_{k=0}^{\infty} (2^k r)^{\alpha-1+s} \phi(2^k r).$$

If we pass to a continuous variable t from the discrete variable k, then we will have

$$|I_\alpha f_2(x)| \le C \|f : M^p_\phi(\mu; s)\| \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} t^{\alpha+s-2} \phi(t) \, dt =$$

$$= C \|f : M^p_\phi(\mu; s)\| \int_r^{\infty} t^{\alpha+s-2} \phi(t) \, dt \le C r^{\alpha+s-1} \phi(r).$$

Here for the last inequality we have used (15). If we apply this pointwise estimate and (16), then we obtain

$$\left(\frac{1}{r^s}\int\limits_B \left|I_\alpha f_2(x)\right|^q d\mu(x)\right)^{1/q} \le C\phi(r)r^{s+\alpha-1}\left\|f : M^p_\phi(\mu;s)\right\| = \\ = C\psi(r)\left\|f : M^p_\phi(\mu;s)\right\|.$$

Consequently,

$$\frac{1}{\psi(r)}\left(\frac{1}{r^s}\int\limits_B \left|I_\alpha f_2(x)\right|^q d\mu(x)\right)^{1/q} \le C\left\|f : M^p_\phi(\mu;s)\right\|. \tag{19}$$

Thus, from (18) and (19)we obtain the boundedness of I_α. □

Remark. If $\alpha + s < 1$, then the condition

$$\int\limits_r^\infty t^{\alpha+s-2}\phi(t)\,dt \le Cr^{\alpha+s-1}\phi(r)$$

follows automatically from the fact that ϕ is almost decreasing. Indeed,

$$\int\limits_r^\infty t^{\alpha+s-2}\phi(t)\,dt \le C\int\limits_r^\infty t^{\alpha+s-2}\phi(r)\,dt = Cr^{\alpha+s-1}\phi(r).$$

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Authors' addresses:

Eridani

Department of Mathematics, Campus C
Airlangga University, Surabaya 60115
Indonesia
E-mail: pakblangkon@yahoo.com, eri.campanato@gmail.com

Yoshihiro Sawano

Department of Mathematics, Kyoto University
Kyoto, 606–8502
Japan
E-mail: yoshihiro-sawano@celery.ocn.ne.jp